

A HOCHSCHILD HOMOLOGY EULER CHARACTERISTIC FOR CIRCLE ACTIONS

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ABSTRACT. We define an “ S^1 -Euler characteristic”, $\tilde{\chi}_{S^1}(X)$, of a circle action on a compact manifold or finite complex X . It lies in the first Hochschild homology group $HH_1(\mathbb{Z}G)$ where G is the fundamental group of X . This $\tilde{\chi}_{S^1}(X)$ is analogous in many ways to the ordinary Euler characteristic. One application is an intuitively satisfying formula for the Euler class (integer coefficients) of the normal bundle to a smooth circle action without fixed points on a manifold. In the special case of a 3-dimensional Seifert fibered space, this formula is particularly effective.

§0. INTRODUCTION

In this paper we introduce the “ S^1 -Euler characteristic” of a finite S^1 -space, i.e., of a topological space X with a given circle action, where X is partitioned as an “ S^1 -CW complex” into finitely many “ S^1 -cells” (see §4). We define the S^1 -Euler characteristic both geometrically and as an algebraic trace and show how it fits into the broad picture of algebraic and differential topology and of algebraic K -theory. A smooth circle action on a compact smooth manifold is known to admit the structure of a finite S^1 -CW complex $[I_1, I_2]$ and therefore has an S^1 -Euler characteristic.

Recall that an S^1 -CW complex X is a union $X = \bigcup_{n \geq 0} \bigcup_j c_j^n$ of its S^1 -cells, c_j^n ; here the j -th S^1 - n -cell c_j^n is equivariantly modeled on $S^1/H_j \times D^n$ where $H_j \leq S^1$ is

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the (closed) isotropy subgroup of the “interior” of the S^1 -cell (see §4 for details). The isotropy group H_j is either finite cyclic in which case we define $|H_j|$ to be the order of H_j , or else $H_j = S^1$ in which case we define $|H_j| = 0$. We will always assume that X is connected. Write G for the fundamental group of X and let $\mathbb{Z}G$ denote the integral groupring of G . Our new invariant lies in the Hochschild homology group $HH_1(\mathbb{Z}G)$. Recall that this abelian group is the first homology of a certain chain complex; 1-chains in this complex are by definition elements of $\mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G$ (see §1 for a detailed account). Thus we write Hochschild 1-chains as finite sums $\sum_j n_j u_j \otimes v_j$ where $u_j, v_j \in G$ and the n_j are integers.

The S^1 -Euler characteristic, denoted by $\tilde{\chi}_{S^1}(X) \in HH_1(\mathbb{Z}G)$, is defined to be the homology class represented by the Hochschild 1-cycle:

$$(0.1) \quad \sum_{n \geq 0} (-1)^{n+1} \sum_j \sum_{i=1}^{|H_j|} g_{j,n} \otimes g_{j,n}^{-1-i}.$$

Here, the element $g_{j,n} \in G$ (dependent on a choice of basepath) has, as a representative loop, an orbit in the interior of the S^1 - n -cell c_j^n traversed once (note that during the action of S^1 , this orbit is traversed $|H_j|$ times if $|H_j| \neq 0$, whereas if $|H_j| = 0$ the orbit is trivial and $g_{j,n} = 1$). It turns out that $\tilde{\chi}_{S^1}(X)$ is independent of basepaths; see §4.

Formula (0.1) for $\tilde{\chi}_{S^1}(X)$ is geometric in the sense that it is described by certain orbits of the S^1 -action. An “Euler characteristic” should have an equivalent and more computable definition as an algebraic trace. We now give such a description of $\tilde{\chi}_{S^1}(X)$.

An S^1 action on a finite CW complex X defines a homotopy $X \times I \rightarrow X$ from id_X , the identity map of X , to itself. This homotopy is not necessarily cellular, but it can be deformed rel $X \times \{0, 1\}$ to a cellular homotopy $F: X \times I \rightarrow X$. Orient the cells of X and choose compatibly oriented lifts in the universal cover, \tilde{X} , of each cell of X . Using cellular chains, let $\tilde{D}: C_*(\tilde{X}) \rightarrow C_*(\tilde{X})$ be the chain homotopy corresponding to the lift $\tilde{F}: \tilde{X} \times I \rightarrow \tilde{X}$ of F with $\tilde{F}_0 = \text{id}_{\tilde{X}}$. Let $\tilde{\partial}: C_*(\tilde{X}) \rightarrow C_*(\tilde{X})$ be the boundary

$\mathbb{Z}G$ -homomorphism. Then we can form the Hochschild 1-chain (which turns out to be a 1-cycle):

$$(0.2) \quad \sum_{i,j} \tilde{\partial}_{ij} \otimes \tilde{D}_{ji}.$$

Here we regard $\tilde{\partial} = (\tilde{\partial}_{ij})$ and $\tilde{D} = (\tilde{D}_{ij})$ as square $\mathbb{Z}G$ -matrices; see Definition 2.1. One of the main theorems of this paper, Theorem 4.9, asserts that the homology classes of the Hochschild 1-cycles (0.1) and (0.2) are essentially the same. We consider the expression (0.2) to be a “trace” because its homology class generalizes the “Dennis trace” of algebraic K -theory (see §2).

One application of the S^1 -Euler characteristic is a formula for the Euler class of the normal bundle to a smooth circle action without fixed points on a smooth closed oriented manifold X . With the above notation, the Poincaré dual of this Euler class turns out to be

$$(0.3) \quad \sum_{n \geq 0} (-1)^n \sum_j \{g_{j,n}\} \in H_1(X; \mathbb{Z})$$

(see Theorem 4.12).¹

As a special case, suppose X is an oriented 3-dimensional Seifert fibered space whose (oriented) quotient surface is Σ . The S^1 -CW structure can then be chosen so that each singular fiber is an S^1 -0-cell. Let $g_{1,0}, \dots, g_{r,0} \in G$ represent the singular fibers and let $\gamma_0 \in G$ represent an ordinary fiber. Formula (0.3) tells us that the Poincaré dual to the Euler class of the normal bundle is

$$(0.4) \quad (\chi(\Sigma) - r)\{\gamma_0\} + \sum_{j=1}^r \{g_{j,0}\} \in H_1(X; \mathbb{Z}).$$

¹Notice the striking analogy between (0.3) and the well known identification of the Poincaré dual of the Euler class of the tangent bundle of a smooth closed oriented manifold Y with the Euler characteristic, $\sum_{n \geq 0} (-1)^n \sum_j \{y_{j,n}\} \in H_0(Y; \mathbb{Z}) \cong \mathbb{Z}$ (here $y_{j,n}$ is a point, thought of as a 0-cycle, in the interior of the j -th n -cell of some finite CW structure on Y so that $\sum_j \{y_{j,n}\}$ is identified with the number of n -cells).

(See Theorem 4.13.) Furthermore, the equivalence of (0.1) and (0.2) implies that (0.4) can be expressed by the computable formula

$$(0.5) \quad A\left(\sum_{i,j} \tilde{\partial}_{ij} D_{ji}\right)$$

where the integer matrix (D_{ij}) is the matrix of the chain homotopy $D: C_*(X) \rightarrow C_*(X)$ (covered by \tilde{D}) which is induced by the homotopy $F: X \times I \rightarrow X$ given by (a cellular approximation to) the circle action, and the homomorphism $A: \mathbb{Z}G \rightarrow G/[G, G] \equiv G_{\text{ab}} \cong H_1(X; \mathbb{Z})$ is the unique extension of the abelianization homomorphism $G \rightarrow G/[G, G]$; see [GN₃].²

Two further results reinforce the analogy between our $\tilde{\chi}_{S^1}(X)$ and the classical Euler characteristic:

- The classical Euler characteristic of a closed odd-dimensional manifold is zero. We show that the image of $\tilde{\chi}_{S^1}(X)$ under the natural homomorphism $HH_1(\mathbb{Z}G) \rightarrow H_1(G)$ (which takes a Hochschild homology class represented by $\sum_i n_i g_i \otimes h_i$ to $A(\sum_i n_i g_i)$) is zero whenever X is a closed even-dimensional smooth S^1 -manifold (see Theorem 4.11 and its Addendum).
- The classical Euler characteristic of a finite CW complex vanishes if the complex admits a circle action with finite isotropy. We show in §5 that the S^1 -Euler characteristic is analogously an obstruction to the existence of an $S^1 \times S^1$ -action with finite isotropy extending a given S^1 -action (Corollary 5.5).

Since $\tilde{\chi}_{S^1}(X)$ lies in the Hochschild homology group $HH_1(\mathbb{Z}G)$, the question naturally arises as to whether it is necessarily contained in the image of the Dennis trace homomorphism $\text{DT}: K_1(\mathbb{Z}G) \rightarrow HH_1(\mathbb{Z}G)$ (see Definition 1.3). Our extensive calculation of $\tilde{\chi}_{S^1}(X)$ in case X is a 3-dimensional Seifert fibered space (§6) shows that this is often *not* true for these examples.

²The formula (0.4) is easily obtained for rational coefficients using the fact that a finite covering of the Seifert fibered space is a circle bundle. To the best of our knowledge, formulas (0.4) and (0.5) with integral coefficients are new.

An “Euler characteristic” should be topologically invariant. In the case of the S^1 –Euler characteristic, the interpretation of this principle is not immediately clear. The precise formulation of topological invariance requires the introduction of some new invariants (presented here in §2) of a somewhat technical nature which extend invariants first developed in [GN₃]. In fact, $\tilde{\chi}_{S^1}(X)$ canonically decomposes into two pieces, the first of which (after the removal of a slight ambiguity) is computed from a homotopy invariant of the underlying space X and the second of which is computed from a *simple* homotopy invariant of X . The connection between §2 and the results discussed in this introduction only becomes clear in §4, the core section of this paper. In §3, we prove a key technical result (Theorem 3.6) needed for our discussion of S^1 –CW complexes in §4. The S^1 –Euler characteristic of an S^1 –complex is formally introduced in §4 (Definition 4.6). Its expression in terms of the topological invariants of §2 is given by Theorem 4.9, and its application to formula (0.3) for the Euler class of the normal bundle to a smooth circle action is given by Theorem 4.12. Formula (0.4) is the content of Theorem 4.13. The vanishing theorems for $\tilde{\chi}_{S^1}(X)$ for a circle action which extend to an $S^1 \times S^1$ –action with finite isotropy are proved in §5. In §6 we compute $\tilde{\chi}_{S^1}(X)$ when X is an oriented 3–dimensional Seifert fibered space (Theorem 6.2) and discuss how much of that structure is detected by $\tilde{\chi}_{S^1}(X)$ (Theorem 6.7). This extended example provides a good illustration of the theory of the S^1 –Euler characteristic. In Appendix A we show how the invariants introduced in §2 give higher order analogs of Gottlieb’s Theorem [Go].

Finally, we remark that the use of tildes in our notation, as in $\tilde{\chi}_{S^1}(X)$, is meant to convey to the reader that the invariant in question is a feature of the universal cover \tilde{X} as opposed to something which can be naively read off from X itself.

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§1. ALGEBRAIC PRELIMINARIES

In this section we review some background material concerning Hochschild homology, traces and Whitehead torsion; some general references for these topics are [C], [I] and [L].

Let R be a commutative ground ring and let S be an associative R -algebra with unit. If M is an S - S bimodule (i.e., a left and right S -module satisfying $(s_1 m)s_2 = s_1(ms_2)$ for all $m \in M$, and $s_1, s_2 \in S$), the *Hochschild chain complex* $\{C_*(S, M), d\}$ consists of $C_n(S, M) = S^{\otimes n} \otimes M$ where $S^{\otimes n}$ is the tensor product of n copies of S and

$$\begin{aligned} d(s_1 \otimes \cdots \otimes s_n \otimes m) &= s_2 \otimes \cdots \otimes s_n \otimes ms_1 \\ &\quad + \sum_{i=1}^{n-1} (-1)^i s_1 \otimes \cdots \otimes s_i s_{i+1} \otimes \cdots \otimes s_n \otimes m \\ &\quad + (-1)^n s_1 \otimes \cdots \otimes s_{n-1} \otimes s_n m. \end{aligned}$$

The tensor products are taken over R . The n -th homology of this complex is the n -th *Hochschild homology of S with coefficient bimodule M* . It is denoted by $HH_n(S, M)$. If $M = S$ with the standard S - S bimodule structure then we usually write $HH_n(S)$ for $HH_n(S, M)$.

We will be concerned mainly with HH_1 and HH_0 which are computed from

$$\begin{array}{ccccc} \cdots & \longrightarrow & S \otimes S \otimes M & \xrightarrow{d} & S \otimes M & \xrightarrow{d} & M \\ & & s_1 \otimes s_2 \otimes m & \mapsto & s_2 \otimes ms_1 - s_1 s_2 \otimes m + s_1 \otimes s_2 m \\ & & & & s \otimes m & \mapsto & ms - sm \end{array}$$

Next, we consider traces in Hochschild homology. If A is a square matrix over M , we interpret its trace $\sum_i A_{ii}$ as an element of M (i.e., as a Hochschild 0-cycle). The corresponding homology class is denoted by $T_0(A) \in HH_0(S, M)$. If A^i , $i = 1, \dots, n$, are

$q_i \times q_{i+1}$ matrices over S and B is a $q_{n+1} \times q_1$ matrix over M , we define $A^1 \otimes \cdots \otimes A^n \otimes B$ to be the $q_1 \times q_1$ matrix with entries in the R -module $S^{\otimes n} \otimes M$ given by

$$(A^1 \otimes \cdots \otimes A^n \otimes B)_{ij} = \sum_{k_2, \dots, k_{n+1}} A_{i, k_2}^1 \otimes A_{k_2, k_3}^2 \otimes \cdots \otimes A_{k_n, k_{n+1}}^n \otimes B_{k_{n+1}, j}.$$

The *trace* of $A^1 \otimes \cdots \otimes A^n \otimes B$, written $\text{trace}(A^1 \otimes \cdots \otimes A^n \otimes B)$, is

$$\sum_{k_1, k_2, \dots, k_{n+1}} A_{k_1, k_2}^1 \otimes A_{k_2, k_3}^2 \otimes \cdots \otimes A_{k_n, k_{n+1}}^n \otimes B_{k_{n+1}, k_1}.$$

which we interpret as a Hochschild n -chain. Observe that the 1-chain $\text{trace}(A \otimes B)$ is a cycle if and only if $\text{trace}(AB) = \text{trace}(BA)$, in which case we denote its homology class by $T_1(A \otimes B) \in HH_1(S, M)$. In this paper S will usually be a groupring over the ground ring R and $M = S$.

We will use the notation G_1 for the set of conjugacy classes of a group G , and $C(g)$ for the conjugacy class of $g \in G$. The partition of G into the union of its conjugacy classes induces a direct sum decomposition of $HH_*(\mathbb{Z}G) \equiv HH_*(\mathbb{Z}G, \mathbb{Z}G)$ as follows: each generating chain $c = g_1 \otimes \cdots \otimes g_n \otimes m$ can be written in *canonical form* as $g_1 \otimes \cdots \otimes g_n \otimes g_n^{-1} \cdots g_1^{-1} g$ where we think of $g = g_1 \cdots g_n m \in G$ as “marking” the conjugacy class $C(g)$. All the generating chains occurring in the boundary $d(c)$ are easily seen to have markers in $C(g)$ when put into canonical form. For $C \in G_1$ let $C_*(\mathbb{Z}G)_C$ be the subgroup of $C_*(\mathbb{Z}G) \equiv C_*(\mathbb{Z}G, \mathbb{Z}G)$ generated by those generating chains whose markers lie in C . The decomposition

$$\mathbb{Z}G \cong \bigoplus_{C \in G_1} \mathbb{Z}C$$

as a direct sum of abelian groups determines a decomposition of chain complexes

$$C_*(\mathbb{Z}G) \cong \bigoplus_{C \in G_1} C_*(\mathbb{Z}G)_C.$$

There results a natural isomorphism $HH_*(\mathbb{Z}G) \cong \bigoplus_{C \in G_1} HH_*(\mathbb{Z}G)_C$ where the summand $HH_*(\mathbb{Z}G)_C$ corresponds to the homology classes of Hochschild cycles marked by the elements of C . We call this summand the C -*component*.

Given any $\mathbb{Z}G$ - $\mathbb{Z}G$ bimodule N let \overline{N} be the left $\mathbb{Z}G$ module whose underlying abelian group is N and whose left module structure is given by $gm = g \cdot m \cdot g^{-1}$. There is a natural isomorphism $\mu_N : HH_*(\mathbb{Z}G, N) \xrightarrow{\cong} H_*(G, \overline{N})$ which is induced from an isomorphism of the Hochschild complex to the bar complex for computing group homology; see [I, Theorem 1.d]. The decomposition $\overline{\mathbb{Z}G} \cong \bigoplus_{C \in G_1} \mathbb{Z}C$ is a direct sum of left $\mathbb{Z}G$ modules, inducing a direct sum decomposition $H_*(G, \overline{\mathbb{Z}G}) \cong \bigoplus_{C \in G_1} H_*(G, \mathbb{Z}C)$. Choosing representatives $g_C \in C$ we have an isomorphism of left $\mathbb{Z}G$ modules $\mathbb{Z}C \cong \mathbb{Z}(G/Z(g_C))$ where $Z(h) = \{g \in G \mid h = ghg^{-1}\}$ denotes the centralizer of $h \in G$. Since $H_*(G, \mathbb{Z}(G/Z(g_C)))$ is naturally isomorphic to $H_*(Z(g_C))$, we obtain a natural isomorphism $HH_*(\mathbb{Z}G) \cong \bigoplus_{C \in G_1} H_*(Z(g_C))$; $HH_*(\mathbb{Z}G)_C$ corresponds to the summand $H_*(Z(g_C))$ under this identification. In particular $HH_0(\mathbb{Z}G) \cong \mathbb{Z}G_1$, the free abelian group generated by the conjugacy classes, and $HH_1(\mathbb{Z}G) \cong \bigoplus_{C \in G_1} H_1(Z(g_C))$, the direct sum of the abelianizations of the centralizers.

Proposition 1.1. *The chain $g \otimes g^{-1}g_C$ is a cycle if and only if $g \in Z(g_C)$. For $g \in Z(g_C)$, the homology class of $g \otimes g^{-1}g_C$ in $HH_1(\mathbb{Z}G)$ corresponds to $\{g\} \in H_1(Z(g_C))$.*

□

The augmentation $\varepsilon : \mathbb{Z}G \rightarrow \mathbb{Z}$ can be viewed as a morphism of $\mathbb{Z}G$ - $\mathbb{Z}G$ bimodules, where \mathbb{Z} is given the trivial bimodule structure, or as a morphism $\varepsilon : \overline{\mathbb{Z}G} \rightarrow \overline{\mathbb{Z}}$ of left $\mathbb{Z}G$ -modules. Then there is an induced chain map $C_*(\mathbb{Z}G, \mathbb{Z}G) \xrightarrow{\varepsilon} C_*(\mathbb{Z}G, \mathbb{Z})$ and a commutative diagram:

$$\begin{array}{ccc} HH_*(\mathbb{Z}G, \mathbb{Z}G) & \xrightarrow{\varepsilon} & HH_*(\mathbb{Z}G, \mathbb{Z}) \\ \mu_{\mathbb{Z}G} \downarrow & & \mu_{\mathbb{Z}} \downarrow \\ H_*(G, \overline{\mathbb{Z}G}) & \xrightarrow{\varepsilon} & H_*(G, \overline{\mathbb{Z}}) \end{array}$$

where the vertical arrows are isomorphisms. Let $\varepsilon_* : HH_*(\mathbb{Z}G) \rightarrow H_*(G)$ denote the resulting homomorphism.

The “abelianization” homomorphism $A: G \rightarrow G_{\text{ab}} = H_1(G)$ extends to a homomorphism of abelian groups $A: \mathbb{Z}G \rightarrow G_{\text{ab}}$. This occurs in the following computation of $\mu_{\mathbb{Z}}$ ([GN₃, Proposition 2.1]):

Proposition 1.2. *If $\sum_i c_i \otimes n_i \in C_1(\mathbb{Z}G, \mathbb{Z})$ is a Hochschild 1-cycle representing $z \in HH_1(\mathbb{Z}G, \mathbb{Z})$, where $c_i \in \mathbb{Z}G$ and $n_i \in \mathbb{Z}$, then $\mu_{\mathbb{Z}}(z) = \sum_i A(c_i n_i) \in H_1(G)$.*

□

We recall the definition of K_1 of a ring. Let $GL(n, R)$ denote the general linear group consisting of all $n \times n$ invertible matrices over R , and let $GL(R)$ be the direct limit of the sequence $GL(1, R) \subset GL(2, R) \subset \cdots$. A matrix in $GL(R)$ is called *elementary* if it coincides with the identity except for a single off-diagonal entry. The subgroup $E(R) \subset GL(R)$ generated by the elementary matrices is precisely the commutator subgroup of $GL(R)$, and the abelian quotient group $GL(R)/E(R)$ is, by definition, $K_1(R)$.

Definition 1.3. The *Dennis trace* homomorphism $DT: K_1(R) \rightarrow HH_1(R)$ is defined as follows. If $\alpha \in K_1(R)$ is represented by an invertible $n \times n$ matrix A then $DT(\alpha) = T_1(A \otimes A^{-1})$ (see [I, Chapter 1]).

In case $R = \mathbb{Z}G$, let $\pm G \subset GL(1, \mathbb{Z}G)$ be the subgroup consisting of 1×1 matrices of the form $[\pm g]$, $g \in G$. The cokernel of the natural homomorphism $\pm G \rightarrow K_1(\mathbb{Z}G)$ is called the *Whitehead group of G* and is denoted by $Wh_1(G)$.

Proposition 1.4. *The image of the composite homomorphism:*

$$\pm G \xrightarrow{i} K_1(\mathbb{Z}G) \xrightarrow{DT} HH_1(\mathbb{Z}G)$$

lies in $HH_1(\mathbb{Z}G)_{C(1)}$.

Proof. For $g \in G$, $DT(i(\pm g)) = \text{homology class of } g \otimes g^{-1} \in HH_1(\mathbb{Z}G)_{C(1)}$. □

We briefly recall how the torsion of an acyclic complex is defined (see [C]). Declare two bases of a finitely generated free right $\mathbb{Z}G$ -module to be *equivalent* if the change of basis

matrix represents an element of $K_1(\mathbb{Z}G)$ which lies in the image of $\pm G \xrightarrow{i} K_1(\mathbb{Z}G)$. Let (C, ∂) be a finitely generated chain complex of right modules over $\mathbb{Z}G$ such that each C_i is free with a given equivalence class of bases. Suppose that C is acyclic. Let $\delta: C \rightarrow C$ be a chain contraction, $C_{\text{odd}} \equiv \oplus_{i \text{ odd}} C_i$ and $C_{\text{even}} \equiv \oplus_{i \text{ even}} C_i$. The restriction of $\partial + \delta$ to C_{odd} is an isomorphism $C_{\text{odd}} \rightarrow C_{\text{even}}$ and so its matrix with respect to bases chosen from each of the given equivalence classes defines an element of $K_1(\mathbb{Z}G)$. The image of this element in $\text{Wh}_1(G)$, denoted $\tau(C)$, is independent of the choice of representatives of the equivalence classes of bases; it is called the *Whitehead torsion* of (C, ∂) .

§2. A REFINED HIGHER EULER CHARACTERISTIC

The goal of this section is to develop topologically invariant refinements of the higher Euler characteristics first introduced in [GN₃]; prior knowledge of [GN₃] is not needed. In §4, these refined invariants will be used to prove our main results about the S^1 -Euler characteristic. In §2(A) the invariants are defined and their basic properties are established. Homotopy and simple homotopy invariance is discussed in §2(B). Their geometric interpretation is given in §2(C) (which might usefully be read in conjunction with §2(A) to provide motivation).

(A) Definition of the invariants.

Let (X, A) be a topological pair where X is a k -space (i.e., a Hausdorff space which has the weak topology determined by the family of its compact subspaces) and $A \subset X$ is closed. Let X^X denote the function space of continuous maps $X \rightarrow X$ with the compact-open topology and let $(X, A)^{(X, A)} \subset X^X$ be the subspace consisting of maps of pairs $(X, A) \rightarrow (X, A)$. Define $\Gamma_{(X, A)} \equiv \pi_1((X, A)^{(X, A)}, \text{id})$. Note that $\Gamma_{(X, A)}$ is abelian because $(X, A)^{(X, A)}$ is an H-space. An element $\gamma \in \Gamma_{(X, A)}$ is represented by a continuous map of pairs $F^\gamma: (X, A) \times I \rightarrow (X, A)$ such that $F_0^\gamma = F_1^\gamma = \text{id}$. When $A = \emptyset$ we write Γ_X for $\Gamma_{(X, \emptyset)}$. The inclusion $(X, A)^{(X, A)} \subset X^X$ induces a homomorphism $\Gamma_{(X, A)} \rightarrow \Gamma_X$.

Suppose X is path connected. Let $x_0 \in X$ be a basepoint and let $G \equiv \pi_1(X, x_0)$. Evaluation at x_0 yields a map $\eta: X^X \rightarrow X$. The image of $\eta_\#: \Gamma_X \rightarrow G$, denoted $\mathcal{G}(X)$, is called the *Gottlieb subgroup* of G . Let $Z(G)$ denote the center of G . It was shown in [Got] that $\mathcal{G}(X) \leq Z(G)$ and that if X is also aspherical then $\mathcal{G}(X) = Z(G)$ and $\eta_\#: \Gamma_X \rightarrow Z(G)$ is an isomorphism. In particular, we may regard $\mathcal{G}(X) \subset G_1$ (where G_1 is the set of conjugacy classes of elements of G). Define

$$\begin{aligned} HH_*(\mathbb{Z}G)' &\equiv \bigoplus_{C \in \mathcal{G}(X)} HH_*(\mathbb{Z}G)_C \\ HH_*(\mathbb{Z}G)'' &\equiv \bigoplus_{C \in G_1 - \mathcal{G}(X)} HH_*(\mathbb{Z}G)_C \end{aligned}$$

There is a left action of $Z(G)$ on $HH_*(\mathbb{Z}G)$. At the level of chains it is defined by

$$\omega \cdot (g_1 \otimes \cdots \otimes g_n \otimes m) = g_1 \otimes \cdots \otimes g_n \otimes (m\omega^{-1})$$

where $\omega \in Z(G)$. One easily checks that this action is compatible with the Hochschild boundary d and hence makes $HH_*(\mathbb{Z}G)$ into a left $Z(G)$ -module. The summand $HH_*(\mathbb{Z}G)_C$ is taken by the left action of ω isomorphically onto the summand $HH_*(\mathbb{Z}G)_{C\omega^{-1}}$ where $C\omega^{-1}$ is the conjugacy class $\{g\omega^{-1} \mid g \in C\}$. Since $\eta_\#$ maps Γ_X into $Z(G)$, $\eta_\#$ defines a left action of Γ_X (and thus also a left action of $\Gamma_{(X,A)}$ via the natural homomorphism $\Gamma_{(X,A)} \rightarrow \Gamma_X$) on $C_*(\mathbb{Z}G, \mathbb{Z}G)$ and on $HH_1(\mathbb{Z}G)$. The group $HH_*(\mathbb{Z}G)$ splits as a direct sum of $\Gamma_{(X,A)}$ -modules:

$$HH_*(\mathbb{Z}G) = HH_*(\mathbb{Z}G)' \oplus HH_*(\mathbb{Z}G)''. \quad$$

Let

$$\pi': HH_*(\mathbb{Z}G) \rightarrow HH_*(\mathbb{Z}G)' \quad \text{and} \quad \pi'': HH_*(\mathbb{Z}G) \rightarrow HH_*(\mathbb{Z}G)''$$

denote the projections. Since the centralizer of any element in $\mathcal{G}(X)$ is all of G , there is a natural isomorphism of $\Gamma_{(X,A)}$ -modules $HH_*(\mathbb{Z}G)' \cong H_1(G) \otimes_{\mathbb{Z}} \mathbb{Z}\mathcal{G}(X)$.

Given a relative CW complex (X, A) , we denote its relative n -skeleton by $(X, A)^n$. Recall (see [Sp]) that the cellular chain complex $C_*(X, A)$ of (X, A) is defined by

$$C_j(X, A) = H_j((X, A)^j, (X, A)^{j-1})$$

with the boundary homomorphism arising from the long exact sequence associated to the triple $((X, A)^j, (X, A)^{j-1}, (X, A)^{j-2})$. Orient the cells of (X, A) , thus establishing a preferred basis for the chain complex $(C_*(X, A), \partial)$.

For the rest of this section we assume that (X, A) is a finite relative CW complex such that:

- (1) A is a k -space.
- (2) X is path connected.
- (3) X has a simply connected universal covering space, \tilde{X} .

Define $\tilde{A} \equiv p^{-1}(A)$ where $p: \tilde{X} \rightarrow X$ is the universal covering projection. The pair (\tilde{X}, \tilde{A}) is a relative CW complex with j -skeleton $(\tilde{X}, \tilde{A})^j = p^{-1}((X, A)^j)$. Choose a lift, \tilde{e} , to the universal cover, \tilde{X} , for each cell e of (X, A) , and orient \tilde{e} compatibly with e . Regard the cellular chain complex $(C_*(\tilde{X}, \tilde{A}), \tilde{\partial})$ as a free right $\mathbb{Z}G$ -module chain complex with preferred basis $\{\tilde{e}\}$; see [GN₁, §1(B)].

By the Cellular Approximation Theorem, each $\gamma \in \Gamma_{(X, A)}$ can be represented by a cellular homotopy $F^\gamma: (X, A) \times I \rightarrow (X, A)$ such that $F_0^\gamma = F_1^\gamma = \text{id}_X$. There is a unique lift, $\tilde{F}^\gamma: (\tilde{X}, \tilde{A}) \times I \rightarrow (\tilde{X}, \tilde{A})$, of F^γ such that $\tilde{F}_0^\gamma = \text{id}_{\tilde{X}}$. Note that \tilde{F}_1^γ is the covering transformation corresponding to the element $\eta_\#(\gamma) \in G$. Let $\tilde{D}_*^\gamma: C_*(\tilde{X}, \tilde{A}) \rightarrow C_{*+1}(\tilde{X}, \tilde{A})$ be the chain homotopy induced by \tilde{F}^γ .

Sign Convention. If \tilde{e} is an oriented k -cell of (\tilde{X}, \tilde{A}) then $\tilde{D}_k^\gamma(\tilde{e})$ is the $(k+1)$ -chain $(-1)^{k+1} \tilde{F}_*^\gamma(\tilde{e} \times I) \in C_{k+1}(\tilde{X}, \tilde{A})$, where $\tilde{e} \times I$ is given the product orientation. This is consistent with the convention that if $E_{i, \epsilon}$ is the face of the cube $I^n = [0, 1]^n$ obtained by

holding the i^{th} coordinate fixed at $\epsilon = 0$ or 1 , then the incidence number $[I^n : E_{i,\epsilon}]$ is $(-1)^{i+\epsilon}$. At the level of cellular n -chains, we have $\partial_n I^n = \sum_{i,\epsilon} [I^n : E_{i,\epsilon}] E_{i,\epsilon}$.

Write $\tilde{\partial} = \bigoplus_k \tilde{\partial}_k$, $\tilde{D}^\gamma = \bigoplus_k (-1)^{k+1} \tilde{D}_k^\gamma$ and $\tilde{I} = \bigoplus_k (-1)^k \text{id}_k$ (viewed as matrices). The chain homotopy relation becomes $\tilde{D}^\gamma \tilde{\partial} - \tilde{\partial} \tilde{D}^\gamma = \tilde{I}(1 - \eta_\#(\gamma)^{-1})$. [Explanation: the minus sign occurs on the left because of the sign convention built into the matrix \tilde{D}^γ ; the right hand side is thus because the 0-end of the homotopy F^γ is lifted to the identity, while the 1-end is lifted to the covering translation corresponding to $\eta_\#(\gamma)$; the inversion occurs because we have G acting on the right.]

The proofs of Propositions 2.5 and 2.2 of [GN₃] carry over directly to show, respectively, that the Hochschild 1-chain $\text{trace}(\tilde{\partial} \otimes \tilde{D}^\gamma)$ is a cycle and that the homology class of this cycle, $T_1(\tilde{\partial} \otimes \tilde{D}^\gamma)$, does not depend on the choice of the cellular map F^γ representing $\gamma \in \Gamma_{(X,A)}$.

Definition 2.1. Define the function $\tilde{\mathcal{X}}_1(X, A): \Gamma_{(X,A)} \rightarrow HH_1(\mathbb{Z}G)$ by:

$$\tilde{\mathcal{X}}_1(X, A)(\gamma) \equiv T_1(\tilde{\partial} \otimes \tilde{D}^\gamma).$$

In case $A = \emptyset$, $\tilde{\mathcal{X}}_1(X, \emptyset) \equiv \tilde{\mathcal{X}}_1(X)$ is identical to the “lift of $\chi_1(X)$ ” defined in [GN₃, §2]. By considering lifts of homotopies, we easily obtain:

Proposition 2.2. *When $HH_1(\mathbb{Z}G)$ is regarded as a left $\Gamma_{(X,A)}$ -module, $\tilde{\mathcal{X}}_1(X, A)$ becomes a derivation; i.e., $\tilde{\mathcal{X}}_1(X, A)(\gamma_1 \gamma_2) = \tilde{\mathcal{X}}_1(X, A)(\gamma_1) + \gamma_1 \cdot \tilde{\mathcal{X}}_1(X, A)(\gamma_2)$. \square*

The dependence of $\tilde{\mathcal{X}}_1(X, A)$ on the choice of oriented cell lifts can be made explicit as follows. Suppose $\mathcal{L}_1 = \{\tilde{e}_j^n\}$ is a choice of oriented cell lifts (which we regard as an ordered basis for $C_*(\tilde{X}, \tilde{A})$). Then any other choice, \mathcal{L}_2 , of oriented cell lifts is of the form $\mathcal{L}_2 = \{\tilde{e}_j^n(\mu_{n,j} g_{n,j})\}$ where $g_{n,j} \in G$ and $\mu_{n,j} = \pm 1$. Let $\tilde{\mathcal{X}}_1(X, A; \mathcal{L}_i)$ be $\tilde{\mathcal{X}}_1(X, A)$ computed with respect to \mathcal{L}_i , $i = 1, 2$. Let U be the diagonal matrix whose diagonal entries are $\mu_{n,j} g_{n,j} \in \pm G \subset \mathbb{Z}G$ (this is the “change of basis matrix”).

Proposition 2.3. For $\gamma \in \Gamma_{(X,A)}$,

$$\tilde{\mathcal{X}}_1(X, A; \mathcal{L}_1)(\gamma) - \tilde{\mathcal{X}}_1(X, A; \mathcal{L}_2)(\gamma) = (1 - \gamma) \cdot T_1(U \otimes U^{-1}).$$

Furthermore, $(1 - \gamma) \cdot T_1(U \otimes U^{-1}) \in HH_1(\mathbb{Z}G)'$.

Proof. By the “change of basis formula”, [GN₁, Proposition 3.3]:

$$\tilde{\mathcal{X}}_1(X, A; \mathcal{L}_1)(\gamma) - \tilde{\mathcal{X}}_1(X, A; \mathcal{L}_2)(\gamma) = T_1(U \otimes U^{-1}(1 - \eta_{\#}(\gamma)^{-1})).$$

Since $\eta_{\#}(\gamma) \in \mathcal{G}(X) \subset Z(G)$,

$$T_1(U \otimes U^{-1}(1 - \eta_{\#}(\gamma)^{-1})) = (1 - \gamma) \cdot T_1(U \otimes U^{-1}).$$

We have $\text{trace}(U \otimes U^{-1}) = \sum_{n,j} g_{n,j} \otimes g_{n,j}^{-1} \in C_1(\mathbb{Z}G, \mathbb{Z}G)_{C(1)}$ and so $(1 - \gamma) \cdot T_1(U \otimes U^{-1}) \in HH_1(\mathbb{Z}G)'$. \square

Definition 2.4. We define derivations:

$$\tilde{\mathcal{X}}'_1(X, A): \Gamma_{(X,A)} \rightarrow HH_1(\mathbb{Z}G)' \quad \text{and} \quad \tilde{\mathcal{X}}''_1(X, A): \Gamma_{(X,A)} \rightarrow HH_1(\mathbb{Z}G)''$$

by $\tilde{\mathcal{X}}'_1(X, A) \equiv \pi' \circ \tilde{\mathcal{X}}_1(X, A)$ and $\tilde{\mathcal{X}}''_1(X, A) \equiv \pi'' \circ \tilde{\mathcal{X}}_1(X, A)$.

The fact that $\tilde{\mathcal{X}}'_1(X, A)$ and $\tilde{\mathcal{X}}''_1(X, A)$ are derivations follows from Proposition 2.2. Of course $\tilde{\mathcal{X}}_1(X, A) = (\tilde{\mathcal{X}}'_1(X, A), \tilde{\mathcal{X}}''_1(X, A))$.

Remark. Proposition 2.3 implies that $\tilde{\mathcal{X}}''_1(X, A)$ does not depend on the choice of oriented cell lifts because $\pi''((1 - \gamma) \cdot T_1(U \otimes U^{-1})) = 0$. A stronger conclusion is found in Theorem 2.10 below.

If M is a left $\Gamma_{(X,A)}$ -module then the cohomology group $H^1(\Gamma_{(X,A)}, M)$ is naturally identified with the quotient, $\text{Der}(\Gamma_{(X,A)}, M) / \text{Inn}(\Gamma_{(X,A)}, M)$, of derivations modulo inner derivations (recall that a derivation $\Gamma_{(X,A)} \rightarrow M$ is inner if it is of the form $\gamma \mapsto (1 - \gamma)m$ for some $m \in M$). We denote the image of a derivation $\Theta: \Gamma_{(X,A)} \rightarrow M$ in $H^1(\Gamma_{(X,A)}, M)$ by $[\Theta]$.

Definition 2.5. We define cohomology classes:

$$\tilde{\chi}_1(X, A) \equiv [\tilde{\mathcal{X}}_1(X, A)] \in H^1(\Gamma_{(X,A)}, HH_1(\mathbb{Z}G))$$

$$\tilde{\chi}'_1(X, A) \equiv [\tilde{\mathcal{X}}'_1(X, A)] \in H^1(\Gamma_{(X,A)}, HH_1(\mathbb{Z}G)') \cong H^1(\Gamma_{(X,A)}, H_1(G) \otimes_{\mathbb{Z}} \mathbb{Z}\mathcal{G}(X))$$

$$\tilde{\chi}''_1(X, A) \equiv [\tilde{\mathcal{X}}''_1(X, A)] \in H^1(\Gamma_{(X,A)}, HH_1(\mathbb{Z}G)'').$$

When $A = \emptyset$, $\tilde{\chi}_1(X, \emptyset) \equiv \tilde{\chi}_1(X)$ is the invariant $\tilde{\chi}_1(X)$ of [GN₃, Proposition 2.7].

From the definitions, it is clear that:

Proposition 2.6. $\tilde{\chi}_1(X, A) = (\tilde{\chi}'_1(X, A), \tilde{\chi}''_1(X, A))$. □

Remark. In the non-aspherical case our invariants can be non-zero even when $\chi(X) \neq 0$ (implying $\mathcal{G}(X)$ is trivial). An example is the real projective plane P^2 . It is shown in [GN₃, §3(D)] that $\tilde{\chi}'_1(P^2) \neq 0$ and $\tilde{\chi}''_1(P^2) \neq 0$.

Some group theoretic and topological consequences of the hypothesis $\tilde{\chi}'_1(X) \neq 0$ are given in Appendix A.

Definition 2.7. The natural homomorphism $\varepsilon_*: HH_1(\mathbb{Z}G) \rightarrow H_1(G)$ (see §1) induces a homomorphism:

$$\varepsilon_*: H^1(\Gamma_{(X,A)}, HH_1(\mathbb{Z}G)) \rightarrow H^1(\Gamma_{(X,A)}, H_1(G)) \cong \text{Hom}(\Gamma_{(X,A)}, H_1(G)).$$

Define three homomorphisms $\Gamma_{(X,A)} \rightarrow H_1(G) = G_{\text{ab}}$:

$$\chi_1(X, A) \equiv \varepsilon_*(\tilde{\chi}_1(X, A)), \quad \chi'_1(X, A) \equiv \varepsilon_*(\tilde{\chi}'_1(X, A)), \quad \chi''_1(X, A) \equiv \varepsilon_*(\tilde{\chi}''_1(X, A)).$$

Note that $\chi_1(X, A) = \chi'_1(X, A) + \chi''_1(X, A)$ by Proposition 2.6.

From Definition 2.1 and Proposition 1.2, we see that

$$\chi_1(X, A)(\gamma) = \sum_{k \geq 0} (-1)^{k+1} A(\text{trace}(\tilde{\partial}_{k+1} D_k^\gamma))$$

where the integer matrix D_k^γ is the matrix of the chain homotopy $C_k(X, A) \rightarrow C_{k+1}(X, A)$ induced by a cellular homotopy $F^\gamma: (X, A) \times I \rightarrow (X, A)$ representing $\gamma \in \Gamma_{(X,A)}$. In case $A = \emptyset$, this is Definition A₁ of [GN₃]. In [GN₃] we also give very different formulations of $\chi_1(X)$ in terms of cap products and of stable homotopy.

(B) Homotopy and simple homotopy invariance.

Suppose that (Y, B) is a finite relative CW complex such that B is a k -space, Y is path connected and has a simply connected universal covering space \tilde{Y} . Let $h: (X, A) \rightarrow (Y, B)$ be a homotopy equivalence of pairs. Choose a homotopy inverse $g: (Y, B) \rightarrow (X, A)$ for h and let $H: (Y, B) \times I \rightarrow (Y, B)$ be a homotopy $hg \simeq \text{id}_Y$ giving a basepath $\hat{H}: I \rightarrow (Y, B)^{(Y,B)}$ from hg to id_Y . The map

$$(X, A)^{(X,A)} \rightarrow (Y, B)^{(Y,B)} \quad f \mapsto h \circ f \circ g$$

together with the basepath \hat{H} induce an isomorphism $h_*: \Gamma_{(X,A)} \rightarrow \Gamma_{(Y,B)}$. Since $\Gamma_{(Y,B)}$ is abelian, h_* is independent of choice of homotopy inverse g and basepath \hat{H} .

Choose $y_0 = h(x_0) \in B$ as the basepoint for Y and let $K = \pi_1(Y, y_0)$. Then h induces an isomorphism $h_\#: G \rightarrow K$. Recall that the *torsion of h* , $\tau(h) \in \text{Wh}_1(K)$, is defined as follows. Let $h': (X, A) \rightarrow (Y, B)$ be a cellular map which is homotopic to h relative to A . Form the mapping cylinder of h'

$$M(h') \equiv X \times I \cup_{h'} Y \equiv (X \times I \sqcup Y) / \sim$$

where “ \sqcup ” is disjoint union and \sim is the equivalence relation generated by $(x, 1) \sim h'(x)$ for $x \in X$. Let $E \equiv A \times I \cup_{h'|_A} B \subset M(h')$. The relative CW complex structures on (X, A) and (Y, B) determine a relative CW complex structure on $(M(h'), E)$; furthermore, the map $i: X \rightarrow M(h')$ given by $x \mapsto (x, 0)$ is the inclusion of a subcomplex. Let $K' = \pi_1(M(h'), (x_0, 0))$. The “collapse” map $c: M(h') \rightarrow Y$ is homotopy equivalence; in particular, it induces an isomorphism $c_\#: K' \rightarrow K$. Let $\widetilde{M}(h')$ be the universal cover

of $M(h')$ and let \tilde{E} be the inverse image of E in $\tilde{M}(h')$. Choose oriented lifts of the cells of $M(h')$ to $\tilde{M}(h')$ (compatible with choice of oriented lifts of cells of X). Since $i_*: C_*(\tilde{X}, \tilde{A}) \rightarrow C_*(\tilde{M}(h'), \tilde{E})$, is a chain homotopy equivalence, the right $\mathbb{Z}K'$ -module complex $\bar{C} \equiv C_*(\tilde{M}(h'), \tilde{E})/i_*(C_*(\tilde{X}, \tilde{A}))$ is acyclic; furthermore, it is free with the evident basis. Let $\tau(\bar{C}) \in \text{Wh}_1(K')$ be the Whitehead torsion of \bar{C} (see §1). Then, by definition, $\tau(h) \equiv c_*(\tau(\bar{C})) \in \text{Wh}_1(K)$.

By Proposition 1.4, the image of the composite homomorphism:

$$\pm K \rightarrow K_1(\mathbb{Z}K) \xrightarrow{\text{DT}} HH_1(\mathbb{Z}K)$$

lies in $HH_1(\mathbb{Z}K)_{C(1)} \subset HH_1(\mathbb{Z}K)'$ and thus $\pi'' \circ \text{DT}$ factors through $\text{Wh}_1(K)$ yielding a homomorphism $\text{DT}'': \text{Wh}_1(K) \rightarrow HH_1(\mathbb{Z}K)''$. The isomorphism $h_\#: G \rightarrow K$ induced by h in turn induces an isomorphism $h_\#: HH_1(\mathbb{Z}G) \rightarrow HH_1(\mathbb{Z}K)$.

Theorem 2.8 (Torsion Formula). *For $\gamma \in \Gamma_{(X,A)}$,*

$$h_\#(\tilde{\mathcal{X}}_1''(X, A)(\gamma)) - \tilde{\mathcal{X}}_1''(Y, B)(h_*(\gamma)) = (1 - h_*(\gamma)) \cdot \text{DT}''(\tau(h)).$$

Proof. Replacing h with the inclusion of X into the mapping cylinder of h , we may assume without loss of generality that $h: (X, A) \hookrightarrow (Y, B)$ is an inclusion of (X, A) into (Y, B) as a subcomplex. Choose oriented lifts of the cells of Y to the universal cover, \tilde{Y} , of Y . Let $\tilde{X} = p^{-1}(X)$ and $\tilde{A} = p^{-1}(A)$ where $p: \tilde{Y} \rightarrow Y$ is the covering projection. Since $h: (X, A) \hookrightarrow (Y, B)$ is a homotopy equivalence, \tilde{X} is the universal cover of X . Given $\gamma \in \Gamma_{(X,A)}$, let $\mu = h_*(\gamma) \in \Gamma_{(Y,B)}$. We can find, using the homotopy extension property, a cellular self homotopy of the identity, $F^\mu: (Y, B) \times I \rightarrow (Y, B)$, representing μ such that $F^\mu((X, A) \times I) \subset (X, A)$ and the restriction of F^μ to $(X, A) \times I$ represents γ . Let $(\tilde{D}_Y^\mu)_*: C_*(\tilde{Y}, \tilde{B}) \rightarrow C_*(\tilde{Y}, \tilde{B})$ be the chain homotopy determined by the lift of F^μ and let $(\tilde{D}_X^\gamma)_*$ be the restriction of $(\tilde{D}_Y^\mu)_*$ to $C_*(\tilde{X}, \tilde{A})$ (this coincides with the chain homotopy associated to the lift of the restriction of F^μ to $(X, A) \times I$). Let $\bar{C} \equiv C_*(\tilde{Y}, \tilde{B})/h_*(C_*(\tilde{X}, \tilde{A}))$

be the quotient chain complex. Then $(\tilde{D}_Y^\mu)_*$ induces a chain homotopy on this complex which we will denote by \bar{D}_*^μ . There is a commutative diagram:

$$\begin{array}{ccccc} C_*(\tilde{X}, \tilde{A}) & \longrightarrow & C_*(\tilde{Y}, \tilde{B}) & \longrightarrow & \bar{C}_* \\ (\tilde{D}_X^\gamma)_* \downarrow & & (\tilde{D}_Y^\mu)_* \downarrow & & \bar{D}_*^\mu \downarrow \\ C_*(\tilde{X}, \tilde{A}) & \longrightarrow & C_*(\tilde{Y}, \tilde{B}) & \longrightarrow & \bar{C}_* \end{array}$$

By [GN₁, Proposition 3.5], we have in $HH_1(\mathbb{Z}K)$:

$$(2.9) \quad T_1(\tilde{\partial}_Y \otimes \tilde{D}_Y^\mu) - T_1(\tilde{\partial}_X \otimes \tilde{D}_X^\gamma) = T_1(\bar{\partial} \otimes \bar{D}^\mu)$$

where $\tilde{\partial}_X$, $\tilde{\partial}_Y$ and $\bar{\partial}$ are the matrices of the boundary operators of $C_*(\tilde{X}, \tilde{A})$, $C_*(\tilde{Y}, \tilde{B})$ and \bar{C}' respectively and \tilde{D}_X^γ , \tilde{D}_Y^μ and \bar{D}^μ are the matrices of $(\tilde{D}_X^\gamma)_*$, $(\tilde{D}_Y^\mu)_*$ and \bar{D}_*^μ respectively.

Since $h: (X, A) \hookrightarrow (Y, B)$ is a homotopy equivalence, \bar{C} is an acyclic chain complex; furthermore, \bar{C} is free with a preferred basis. In particular, the torsion $\tau(\bar{C}) \in \text{Wh}_1(K)$ is defined. By [GN₁, Proposition 3.7]:

$$T_1(\bar{\partial} \otimes \bar{D}^\mu) = -T_1(V \otimes V^{-1}(1 - \eta_\#(\mu)^{-1}))$$

where V is an invertible matrix representing $\tau(\bar{C}) = \tau(h)$ (see the proof of [GN₁, Proposition 7.1]). We have

$$T_1(V \otimes V^{-1}(1 - \eta_\#(\mu)^{-1})) = (1 - \mu) \cdot T_1(V \otimes V^{-1}) = (1 - \mu) \cdot \text{DT}(b)$$

where $b \in K_1(\mathbb{Z}K)$ is represented by V . Also

$$T_1(\tilde{\partial}_Y \otimes \tilde{D}_Y^\mu) = \tilde{\mathcal{X}}_1(Y, B)(\mu) \quad \text{and} \quad T_1(\tilde{\partial}_X \otimes \tilde{D}_X^\gamma) = \tilde{\mathcal{X}}_1(X, A)(\gamma).$$

Substituting into (2.9) and applying $\pi'': HH_1(\mathbb{Z}K) \rightarrow HH_1(\mathbb{Z}K)''$ yields the conclusion.

□

We say that a homotopy equivalence of pairs $h: (X, A) \rightarrow (Y, B)$ is *simple* if $\tau(h) = 0$ and thus Theorem 2.8 yields:

Theorem 2.10 (Simple homotopy invariance). *The derivation*

$$\tilde{\mathcal{X}}_1''(X, A) : \Gamma_{(X, A)} \rightarrow HH_1(\mathbb{Z}G)''$$

is an invariant of the simple homotopy type of the pair (X, A) , i.e., if $h : (X, A) \rightarrow (Y, B)$ is a simple homotopy equivalence then the diagram

$$\begin{array}{ccc} \Gamma_{(X, A)} & \xrightarrow{\tilde{\mathcal{X}}_1''(X, A)} & HH_1(\mathbb{Z}G) \\ h_* \downarrow & & \downarrow h_\# \\ \Gamma_{(Y, B)} & \xrightarrow{\tilde{\mathcal{X}}_1''(Y, B)} & HH_1(\mathbb{Z}K) \end{array}$$

commutes. □

Remark. We suspect that the derivation $\tilde{\mathcal{X}}_1''(X, A)$ is *not* an invariant of the homotopy type of the pair (X, A) . This would be the case if one could find an example of a homotopy equivalence $h : (X, A) \rightarrow (Y, B)$ such that the term $(1 - h_*(\gamma)) \cdot DT''(\tau(h))$ is not identically zero.

A *subdivision* (X', A) of a relative CW complex (X, A) is another relative CW structure on the underlying topological pair (X, A) such that each open cell of (X', A) is contained in some open cell of (X, A) . The identity $(X, A) \rightarrow (X', A)$ is a cellular map and a simple homotopy equivalence. Hence:

Corollary 2.11. $\tilde{\mathcal{X}}_1''(X, A) = \tilde{\mathcal{X}}_1''(X', A)$, i.e., $\tilde{\mathcal{X}}_1''(X, A)$ is invariant under subdivision.

□

The cohomology classes represented by the derivations $\tilde{\mathcal{X}}_1(X, A)$, $\tilde{\mathcal{X}}_1'(X, A)$ and $\tilde{\mathcal{X}}_1''(X, A)$ have a stronger invariance property:

Theorem 2.12 (Homotopy Invariance). *The cohomology classes $\tilde{\chi}_1(X, A)$, $\tilde{\chi}_1'(X, A)$ and $\tilde{\chi}_1''(X, A)$ are invariants of the homotopy type of (X, A) , i.e., if $h : (X, A) \rightarrow (Y, B)$*

is a homotopy equivalence then $h^* \tilde{\chi}_1(Y, B) = \tilde{\chi}_1(X, A)$, $h^* \tilde{\chi}'_1(Y, B) = \tilde{\chi}'_1(X, A)$ and $h^* \tilde{\chi}''_1(Y, B) = \tilde{\chi}''_1(X, A)$.

Proof. The proof given in [GN₃, Theorem 2.9] carries over directly. Alternatively, equation (2.9) decomposes into a pair equations via the decomposition of $HH_1(\mathbb{Z}K)$ into $HH_1(\mathbb{Z}K)' \oplus HH_1(\mathbb{Z}K)''$. The conclusion can then be deduced from the observation that, in the proof of Theorem 2.8, the derivation $\mu \mapsto T_1(\bar{\partial} \otimes \bar{D}^\mu) = (1 - \mu) \cdot \text{DT}(b)$ is inner. \square

We can now define $\tilde{\chi}_1(N)$, $\tilde{\chi}'_1(N)$, $\tilde{\chi}''_1(N)$ and $\tilde{\mathcal{X}}''_1(N)$ for a space N which is homeomorphic to a compact polyhedron:

Definition 2.13. Let N be a path connected topological space which is homeomorphic to a compact polyhedron. Let $G = \pi_1(N, x_0)$. We define a derivation

$$\tilde{\mathcal{X}}''_1(N): \Gamma_N \rightarrow HH_1(\mathbb{Z}G)'' \quad \text{by} \quad \tilde{\mathcal{X}}''_1(N) \equiv h_\# \circ \tilde{\mathcal{X}}''_1(|K|) \circ (h^{-1})_*$$

where K is a finite simplicial complex and $h: |K| \rightarrow N$ is a homeomorphism. We also define cohomology classes by:

$$\begin{aligned} \tilde{\chi}_1(N) &\equiv (h^{-1})^* \tilde{\chi}_1(|K|) \in H^1(\Gamma_N, HH_1(\mathbb{Z}G)) \\ \tilde{\chi}'_1(N) &\equiv (h^{-1})^* \tilde{\chi}'_1(|K|) \in H^1(\Gamma_N, HH_1(\mathbb{Z}G)') \\ \tilde{\chi}''_1(N) &\equiv (h^{-1})^* \tilde{\chi}''_1(|K|) \in H^1(\Gamma_N, HH_1(\mathbb{Z}G)''). \end{aligned}$$

The homomorphisms $\chi_1(N)$, $\chi'_1(N)$ and $\chi''_1(N)$ are similarly defined:

$$\begin{aligned} \chi_1(N) &\equiv (h^{-1})^* \chi_1(|K|) \in \text{Hom}(\Gamma_N, H_1(G)) \\ \chi'_1(N) &\equiv (h^{-1})^* \chi'_1(|K|) \in \text{Hom}(\Gamma_N, H_1(G)) \\ \chi''_1(N) &\equiv (h^{-1})^* \chi''_1(|K|) \in \text{Hom}(\Gamma_N, H_1(G)). \end{aligned}$$

Note that $\chi_1(N) = \varepsilon_*(\tilde{\chi}_1(N))$, $\chi'_1(N) = \varepsilon_*(\tilde{\chi}'_1(N))$ and $\chi''_1(N) = \varepsilon_*(\tilde{\chi}''_1(N))$.

By Theorem 2.10, and Chapman's Theorem that homeomorphisms of compact polyhedra are simple homotopy equivalences [Ch], the derivation $\tilde{\mathcal{X}}_1''(N)$ is independent of the choice of $h: |K| \rightarrow N$. Also, by Theorem 2.12, the cohomology classes $\tilde{\chi}_1(N)$, $\tilde{\chi}'_1(N)$ and $\tilde{\chi}''_1(N)$ are independent of h .

We end this subsection with a technical result, Proposition 2.14, which will be used in §3.

Suppose the topological pair $(Z, C) \subset (X, A)$, where C is closed in A , is a subcomplex of the finite relative CW complex (X, A) (i.e., (Z, C) is endowed with the structure of a relative CW complex such that each open cell of (Z, C) is an open cell of (X, A)). Assume Z is non-empty, path connected and has a simply connected universal covering space \bar{Z} .

Define $\bar{C} = \bar{p}^{-1}(C)$ where $\bar{p}: \bar{Z} \rightarrow Z$ be the universal covering projection. Let $\tilde{Z} = p^{-1}(Z) \subset \tilde{X}$ and $\tilde{C} = p^{-1}(C) \subset \tilde{X}$. Let $H = \pi_1(Z, z_0)$ where $z_0 \in Z$ is a chosen basepoint for Z . Let $\sigma: I \rightarrow X$ be a basepath from z_0 to x_0 (the basepoint of X). The inclusion $i: Z \hookrightarrow X$ and the basepath σ determine a homomorphism $i_\# : H \rightarrow G$. Let $\bar{P}: \bar{Z} \rightarrow \tilde{Z}$ be a lift of \bar{p} (i.e., $p \circ \bar{P} = i \circ \bar{p}$). Since \bar{P} is cellular, it induces a homomorphism of cellular chain complexes $\bar{P}_*: C_*(\bar{Z}, \bar{C}) \rightarrow C_*(\tilde{Z}, \tilde{C})$. Give $C_*(\bar{Z}, \bar{C}) \otimes_{i_\#} \mathbb{Z}G$ the right G -module structure $(u \otimes g') \cdot g = u \otimes (g'g)$. Then \bar{P}_* induces an isomorphism $\bar{P}'_*: C_*(\bar{Z}, \bar{C}) \otimes_{i_\#} \mathbb{Z}G \rightarrow C_*(\tilde{Z}, \tilde{C})$ of right G -module chain complexes given by $\bar{P}'_*(u \otimes g) = \bar{P}_*(u)g$. Choose oriented lifts of the cells of (Z, C) to (\bar{Z}, \bar{C}) . These lifts and the map \bar{P} determine oriented lifts of the cells of (Z, C) (regarded as a subcomplex of (X, A)) to (\tilde{Z}, \tilde{C}) . Choose oriented lifts of the remaining cells of (X, A) to (\tilde{X}, \tilde{A}) .

Suppose $F^\gamma: (X, A) \times I \rightarrow (X, A)$ is a cellular homotopy such that $F_0^\gamma = F_1^\gamma = \text{id}$ and $F^\gamma((Z, C) \times I) \subset (Z, C)$. Let $\bar{F}_{(Z, C)}^\gamma: (\bar{Z}, \bar{C}) \times I \rightarrow (\bar{Z}, \bar{C})$ be the lift of the restriction of F^γ to $(Z, C) \times I$ such that $(\bar{F}_{(Z, C)}^\gamma)_0 = \text{id}$. Let $\tilde{F}_{(Z, C)}^\gamma: (\tilde{Z}, \tilde{C}) \times I \rightarrow (\tilde{Z}, \tilde{C})$ be the restriction of the lift $\tilde{F}^\gamma: (\tilde{X}, \tilde{A}) \times I \rightarrow (\tilde{X}, \tilde{A})$ of F^γ such that $(\tilde{F}^\gamma)_0 = \text{id}$. Let $\bar{D}_*^\gamma: C_*(\bar{Z}, \bar{C}) \rightarrow C_*(\bar{Z}, \bar{C})$ and $\tilde{D}_*^\gamma: C_*(\tilde{Z}, \tilde{C}) \rightarrow C_*(\tilde{Z}, \tilde{C})$ be the chain homotopies determined by $\bar{F}_{(Z, C)}^\gamma$

and $\tilde{F}_{(Z,C)}^\gamma$ respectively. Since $\bar{P}'_* \circ (\bar{D}_*^\gamma \otimes \text{id}) = \tilde{D}_*^\gamma$, we have $i_\#(\bar{D}^\gamma) = \tilde{D}^\gamma$ and $i_\#(\bar{\partial}) = \tilde{\partial}$ (here \bar{D}^γ and $\bar{\partial}$ are matrices over $\mathbb{Z}H$, \tilde{D}^γ and $\tilde{\partial}$ are matrices over $\mathbb{Z}G$, $\bar{\partial}$ is the matrix of the boundary operator of $C_*(\bar{Z}, \bar{C})$ and $\tilde{\partial}$ is the matrix of the boundary operator of $C_*(\tilde{Z}, \tilde{C})$). Thus we obtain the following proposition which will be useful in §3:

Proposition 2.14. *For $(Z, C) \subset (X, A)$ as above, $i_*(\text{trace}(\bar{\partial} \otimes \bar{D}^\gamma)) = \text{trace}(\tilde{\partial} \otimes \tilde{D}^\gamma)$.*

□

(C) Geometric Interpretation.

Let X be a compact oriented smooth (or piecewise linear) manifold. In cases of interest, $\mathcal{G}(X)$ is non-trivial and so $\chi(X) = 0$ by Gottlieb's Theorem [Got]. Thus, F^γ can be perturbed to a map \bar{F}^γ whose fixed point set, $\text{Fix}(\bar{F}^\gamma) = \{(x, t) \in X \times I \mid \bar{F}^\gamma(x, t) = x\}$, consists of $X \times \{0, 1\}$ together with finitely many naturally oriented circles lying in $\overset{\circ}{X} \times (\epsilon, 1 - \epsilon)$ for some $\epsilon > 0$. Two such circles V_1 and V_2 lie *in the same fixed point class* if for some $(x_i, t_i) \in V_i$ there is a path ν in $X \times I$ from (x_1, t_1) to (x_2, t_2) such that $(p \circ \nu)(\bar{F}^\gamma \circ \nu)^{-1}$ is homotopically trivial where $p: X \times I \rightarrow X$ denotes projection. Using $(x_0, 0)$ as the basepoint, let μ be a path from $(x_0, 0)$ to a point (x, t) lying in the circle of fixed points V . We associate with V the conjugacy class C of the “marker” $g_C \equiv [(p \circ \mu)(\bar{F}^\gamma \circ \mu)^{-1}] \in G$; this conjugacy class is independent of the choice of μ . Let ω be the loop based at (x, t) which traverses V once in the direction of its orientation. Then $[p \circ (\mu\omega\mu^{-1})] = h$ lies in $Z(g_C)$. In this way, we also associate with V an element of $H_1(Z(g_C)) \cong (Z(g_C))_{\text{ab}}$. If there are two circles V_1 and V_2 in the same fixed point class, we reach the same centralizer $Z(g_C)$ from both circles provided the path used for $(x_1, t_1) \in V_1$ is μ , and the path used for $(x_2, t_2) \in V_2$ is $\mu\nu$, where ν is such that $(p \circ \nu)(\bar{F}^\gamma \circ \nu)^{-1}$ is homotopically trivial. One treats any (finite) number of circles similarly.

We form $\hat{F}^\gamma: X \times [-\epsilon, 1 - \epsilon] \rightarrow X$ by setting

$$\hat{F}^\gamma(x, t) = \begin{cases} \bar{F}^\gamma(x, t) & \text{if } t \in [0, 1 - \epsilon] \\ \bar{F}^\gamma(x, 1 + t) & \text{if } t \in [-\epsilon, 0]. \end{cases}$$

Perturbing \hat{F}^γ near $X \times \{0\}$, we may assume that $\text{Fix}(\hat{F}^\gamma)$ is a union of naturally oriented circles: the previous circles and some new ones near $X \times \{0\}$. With each *new* circle we associate the trivial conjugacy class and an element $[p \circ (\mu\omega\mu^{-1})]$, as before, lying in $H_1(G) \equiv H_1(Z(1))$.

In this way we obtain from F^γ a slightly altered map \hat{F}^γ , and we associate with $\text{Fix}(\hat{F}^\gamma)$ an element of $\bigoplus_{C \in G_1} H_1(Z(g_C)) \cong HH_1(\mathbb{Z}G)$; for details see [GN₂; §1], [GN₃; §1], and [GN₁; §6].

If we only consider (oriented) circles whose associated conjugacy classes lie $\mathcal{G}(X)$ [respectively $G_1 - \mathcal{G}(X)$] we get geometric interpretations of $\tilde{\chi}'_1(X)(\gamma)$ [respectively $\tilde{\chi}''_1(X)(\gamma)$]. By interpreting these oriented circles as homology classes in $X \times I$, using $H_1(X \times I) \cong H_1(X) \cong H_1(G)$, we get $\chi'_1(X)(\gamma)$ and $\chi''_1(X)(\gamma)$.

§3. FILTERED CELL COMPLEXES

The appropriate notion of a “ G -CW complex” [I₁], where G is a compact Lie group, requires us to consider spaces obtained by attaching cells which a priori are more general than CW-complexes. In this paper we will only be concerned with the cases $G = S^1$, the circle group, and $G = T^2 \equiv S^1 \times S^1$, the 2-torus. The main result of this section is Theorem 3.6 which will be used in §4 (see Theorem 4.9) to establish the connection between the topological invariants of §2 and the S^1 -Euler characteristic.

Definition 3.1. A *filtered cell complex* is a compact Hausdorff space X together with a filtration $\emptyset \equiv X_{-1} \subset X_0 \subset \cdots \subset X_N = X$ by closed subsets such that for each $n = 0, \dots, N$ the pair (X_n, X_{n-1}) is a finite relative CW complex.

Note that X_0 is a finite CW complex.

A filtered cell complex is a compact ANR [Wh]; however, as we haven't fully developed 1-parameter fixed point theory for general compact ANR's, it is convenient to deal with a subclass of such spaces:

Definition 3.2. A filtered cell complex X is *polyhedral* if X_0 is a compact polyhedron and all attaching maps in the relative CW complexes (X_n, X_{n-1}) are piecewise linear.

In particular, X is itself a polyhedron.

Given a filtered cell complex X with filtration $\mathcal{F} = \{X_i\}$, let $(X; \mathcal{F})^{(X; \mathcal{F})}$ denote the subspace of X^X consisting of filtration preserving maps, i.e., maps $f: X \rightarrow X$ such that $f(X_i) \subset X_i$ for all i . Define $\Gamma_{(X; \mathcal{F})} \equiv \pi_1((X; \mathcal{F})^{(X; \mathcal{F})}, \text{id})$. For $j \geq i$, restriction to X_j yields a map $(X; \mathcal{F})^{(X; \mathcal{F})} \rightarrow (X_j, X_i)^{(X_j, X_i)}$ which in turn induces a homomorphism $\Gamma_{(X; \mathcal{F})} \rightarrow \Gamma_{(X_j, X_i)}$. We abuse notation: given $\gamma \in \Gamma_{(X; \mathcal{F})}$ we also denote by γ the image of γ in $\Gamma_{(X_j, X_i)}$.

By an inductive application of the relative Cellular Approximation Theorem and the homotopy extension property, any $\gamma \in \Gamma_{(X; \mathcal{F})}$ can be represented by a homotopy $F^\gamma: X \times I \rightarrow X$ such that:

- (1) $F_0^\gamma = F_1^\gamma = \text{id}$.
- (2) F^γ is filtration preserving, i.e., $F^\gamma(X_n \times I) \subset X_n$ for all n .
- (3) For each n , the map of pairs $F^{\gamma, n, n-1}: (X_n, X_{n-1}) \times I \rightarrow (X_n, X_{n-1})$ given by the restriction of F^γ to X_n is cellular.

Suppose X is path connected. Let \tilde{X}_n be the inverse image of X_n in \tilde{X} , the universal cover of X . Let $G \equiv \pi_1(X, x_0)$. For each n and for each cell e of (X_n, X_{n-1}) , choose an oriented lift of e to \tilde{X} . Let $\tilde{F}^\gamma: \tilde{X} \times I \rightarrow \tilde{X}$ be the unique lift of F^γ such that $\tilde{F}_0^\gamma = \text{id}$ (recall that \tilde{F}_1^γ is the covering transformation corresponding to $\eta_\#(\gamma) \in G$). Since $\tilde{F}^{\gamma, n, n-1}: (\tilde{X}_n, \tilde{X}_{n-1}) \times I \rightarrow (\tilde{X}_n, \tilde{X}_{n-1})$ given by the restriction of \tilde{F}^γ to \tilde{X}_n is

cellular, we obtain chain homotopies (retaining the Sign Convention of §2):

$$\tilde{D}_*^{\gamma, n, n-1}: C_*(\tilde{X}_n, \tilde{X}_{n-1}) \rightarrow C_*(\tilde{X}_n, \tilde{X}_{n-1}) \quad n \geq 0.$$

Let $\tilde{\partial}_*^{n, n-1}$ denote the boundary operator in $C_*(\tilde{X}_n, \tilde{X}_{n-1})$.

Definition 3.3. Define the function $\tilde{\mathcal{X}}_1(X; \mathcal{F}): \Gamma_{(X; \mathcal{F})} \rightarrow HH_1(\mathbb{Z}G)$ by

$$\tilde{\mathcal{X}}_1(X; \mathcal{F})(\gamma) \equiv \sum_{n \geq 0} T_1(\tilde{\partial}^{n, n-1} \otimes \tilde{D}^{\gamma, n, n-1}).$$

It is straightforward to verify that, for a given choice of lifts of cells, $\tilde{\mathcal{X}}_1(X; \mathcal{F})(\gamma)$ does not depend on the choice of F^γ representing γ and that the function $\tilde{\mathcal{X}}_1(X; \mathcal{F})$ is a derivation. As in §2, we define derivations:

$$\tilde{\mathcal{X}}'_1(X; \mathcal{F}): \Gamma_{(X; \mathcal{F})} \rightarrow HH_1(\mathbb{Z}G)' \quad \text{and} \quad \tilde{\mathcal{X}}''_1(X; \mathcal{F}): \Gamma_{(X; \mathcal{F})} \rightarrow HH_1(\mathbb{Z}G)''$$

by $\tilde{\mathcal{X}}'_1(X; \mathcal{F}) \equiv \pi' \circ \tilde{\mathcal{X}}_1(X; \mathcal{F})$ and $\tilde{\mathcal{X}}''_1(X; \mathcal{F}) \equiv \pi'' \circ \tilde{\mathcal{X}}_1(X; \mathcal{F})$. Observe that $\tilde{\mathcal{X}}_1(X; \mathcal{F}) = (\tilde{\mathcal{X}}'_1(X; \mathcal{F}), \tilde{\mathcal{X}}''_1(X; \mathcal{F}))$.

The term $T_1(\tilde{\partial}^{n, n-1} \otimes \tilde{D}^{\gamma, n, n-1})$ appearing in Definition 3.3 can be identified as follows. Let $X_{n,1}, \dots, X_{n,k_n}$ be the path components of X_n . For $j = 1, \dots, k_n$, choose basepoints $x_{n,j} \in X_{n,j}$ and basepaths $\sigma_{n,j}: I \rightarrow X$ from $x_{n,j}$ to x_0 (the basepoint for X). Let $H_{n,j} \equiv \pi_1(X_{n,j}, x_{n,j})$ and let $i_j: H_{n,j} \rightarrow G$ the homomorphism determined by the inclusion $X_{n,j} \subset X$ and the basepath $\sigma_{n,j}$. By Proposition 2.14, we have (after choosing cell lifts in the manner prescribed in the discussion preceding that proposition):

Proposition 3.4. For $n \geq 0$:

$$T_1(\tilde{\partial}^{n, n-1} \otimes \tilde{D}^{\gamma, n, n-1}) = \sum_{j=1}^{k_n} (i_j)_* (\tilde{\mathcal{X}}_1(X_{n,j}, X_{n,j} \cap X_{n-1})(\gamma)). \quad \square$$

Proposition 3.5. *Suppose X is a finite connected CW complex and $\mathcal{F} = \{X_i\}$ is a filtration of X by subcomplexes. Then for $\gamma \in \Gamma_{(X;\mathcal{F})}$, $\tilde{\mathcal{X}}_1(X)(\gamma) = \tilde{\mathcal{X}}_1(X;\mathcal{F})(\gamma)$.*

Proof. Clearly, X together with \mathcal{F} is a filtered cell complex. Recall that $\tilde{X}_n = p^{-1}(X_n)$ where $p: \tilde{X} \rightarrow X$ is the universal covering projection. Choose oriented lifts of each cell of X to \tilde{X} . Let \tilde{X}_n^k be the k -skeleton of \tilde{X}_n . The Mayer-Vietoris sequence yields a short exact sequence:

$$0 \rightarrow H_k(\tilde{X}_{n-1}^k, \tilde{X}_{n-1}^{k-1}) \rightarrow H_k(\tilde{X}_n^k, \tilde{X}_n^{k-1}) \rightarrow H_k(\tilde{X}_n^k \cup \tilde{X}_{n-1}, \tilde{X}_n^{k-1} \cup \tilde{X}_{n-1}) \rightarrow 0$$

and thus for each $n \geq 0$ we have an exact sequence of cellular chain complexes:

$$0 \rightarrow C_*(\tilde{X}_{n-1}) \rightarrow C_*(\tilde{X}_n) \rightarrow C_*(\tilde{X}_n, \tilde{X}_{n-1}) \rightarrow 0.$$

Represent $\gamma \in \Gamma_{(X;\mathcal{F})}$ by a cellular map $F^\gamma: X \times I \rightarrow X$ such that $F_0^\gamma = F_1^\gamma = \text{id}$ and $F^\gamma(X_n \times I) \subset X_n$ for all n . Let $\tilde{F}^{\gamma,n}: \tilde{X}_n \times I \rightarrow \tilde{X}_n$ denote the restriction of $\tilde{F}^\gamma: \tilde{X} \times I \rightarrow \tilde{X}$ to \tilde{X}_n and let $\tilde{D}_*^{\gamma,n}: C_*(\tilde{X}_n) \rightarrow C_*(\tilde{X}_n)$ be the corresponding chain homotopy. Let $\tilde{\partial}_*^n$ denote the boundary operator in $C_*(\tilde{X}_n)$. There is a commutative diagram:

$$\begin{array}{ccccc} C_*(\tilde{X}_{n-1}) & \longrightarrow & C_*(\tilde{X}_n) & \longrightarrow & C_*(\tilde{X}_n, \tilde{X}_{n-1}) \\ \tilde{D}_*^{\gamma,n-1} \downarrow & & \tilde{D}_*^{\gamma,n} \downarrow & & \tilde{D}_*^{\gamma,n,n-1} \downarrow \\ C_*(\tilde{X}_{n-1}) & \longrightarrow & C_*(\tilde{X}_n) & \longrightarrow & C_*(\tilde{X}_n, \tilde{X}_{n-1}). \end{array}$$

By [GN₁, Proposition 3.5],

$$T_1(\tilde{\partial}^{n,n-1} \otimes \tilde{D}^{\gamma,n,n-1}) = T_1(\tilde{\partial}^n \otimes \tilde{D}^{\gamma,n}) - T_1(\tilde{\partial}^{n-1} \otimes \tilde{D}^{\gamma,n-1}).$$

Substituting into Definition 3.3,

$$\tilde{\mathcal{X}}_1(X;\mathcal{F})(\gamma) = \sum_{n \geq 0} T_1(\tilde{\partial}^{n,n-1} \otimes \tilde{D}^{\gamma,n,n-1}) = \sum_{n \geq 0} (T_1(\tilde{\partial}^n \otimes \tilde{D}^{\gamma,n}) - T_1(\tilde{\partial}^{n-1} \otimes \tilde{D}^{\gamma,n-1})).$$

The last sum collapses to

$$T_1(\tilde{\partial}^N \otimes \tilde{D}^{\gamma,N}) = T_1(\tilde{\partial} \otimes \tilde{D}^\gamma) = \tilde{\mathcal{X}}_1(X)(\gamma)$$

where $X_N = X$. □

The following theorem is the main technical ingredient in the proof of Theorem 4.9 where it will be applied to an S^1 -CW complex which is filtered by its S^1 -skeleta.

Theorem 3.6. *Let X be a path connected polyhedral filtered cell complex with filtration \mathcal{F} . Then $\tilde{\mathcal{X}}_1''(X) \circ i = \tilde{\mathcal{X}}_1''(X; \mathcal{F})$ and $i^* \tilde{\chi}_1'(X) = [\tilde{\mathcal{X}}_1'(X; \mathcal{F})]$ where $i: \Gamma_{(X; \mathcal{F})} \rightarrow \Gamma_X$ is the natural homomorphism.*

Proof. Since the filtered cell complex $(X; \mathcal{F})$ is polyhedral (see Definition 3.2), X is a compact polyhedron and there is a triangulation X' of X and a filtration $\mathcal{F}' \equiv \{X'_n\}$ of X' by subcomplexes such that for each n , (X'_n, X_{n-1}) is a subdivision of the relative CW complex (X_n, X_{n-1}) . By Proposition 3.5, $\tilde{\mathcal{X}}_1(X') \circ i = \tilde{\mathcal{X}}_1(X'; \mathcal{F}')$.

With notation as in Proposition 3.4, for each $n \geq 0$ let $X_{n,j}$, $j = 1, \dots, k_n$ be the set of path components of X_n ; let $X'_{n,j}$ be the triangulation of $X_{n,j}$ determined by X'_n . The proof of Theorem 2.8 shows that for $\gamma \in \Gamma_{(X; \mathcal{F})}$:

$$\tilde{\mathcal{X}}_1(X_{n,j}, X_{n,j} \cap X_{n-1})(\gamma) - \tilde{\mathcal{X}}_1(X'_{n,j}, X_{n,j} \cap X_{n-1})(\gamma) = (1 - \gamma) \cdot \text{DT}(b_{n,j})$$

where $b_{n,j} \in K_1(\mathbb{Z}H_{n,j})$ is a representative of the torsion of the identity map $\text{id}_{n,j}: (X_{n,j}, X_{n,j} \cap X_{n-1}) \rightarrow (X'_{n,j}, X_{n,j} \cap X_{n-1})$. Each $\text{id}_{n,j}$ is a simple homotopy equivalence and thus $\text{DT}(b_{n,j}) \in HH_1(\mathbb{Z}H_{n,j})_{C(1)}$. It follows that $(1 - \gamma) \cdot (i_j)_* \text{DT}(b_{n,j}) \in HH_1(\mathbb{Z}G)'$. By Proposition 3.4,

$$\tilde{\mathcal{X}}_1(X; \mathcal{F})(\gamma) - \tilde{\mathcal{X}}_1(X'; \mathcal{F}')(\gamma) = (1 - \gamma) \cdot \left(\sum_{n,j} (i_j)_* \text{DT}(b_{n,j}) \right).$$

From this we deduce that $\tilde{\mathcal{X}}_1''(X; \mathcal{F}) = \tilde{\mathcal{X}}_1''(X'; \mathcal{F}')$ and that $[\tilde{\mathcal{X}}_1'(X; \mathcal{F})] = [\tilde{\mathcal{X}}_1'(X'; \mathcal{F}')]$. Now by Definition 2.13, $\tilde{\mathcal{X}}_1''(X) = \tilde{\mathcal{X}}_1''(X')$ and $\tilde{\chi}_1'(X) = \tilde{\chi}_1'(X')$. □

§4. THE S^1 -EULER CHARACTERISTIC OF AN S^1 -CW COMPLEX

In this section we introduce the “ S^1 -Euler characteristic” (Definition 4.6) of a finite S^1 -CW complex X and show in Proposition 4.8 that it coincides with $\tilde{\chi}_1(X; \mathcal{F})(\gamma)$ where $\gamma \in \Gamma_X$ is the element defined by the S^1 -action and \mathcal{F} is the filtration of X by S^1 -skeleta. This, together with Theorem 3.6, yields Theorem 4.9 which establishes the precise relationship between the S^1 -Euler characteristic and the topological invariants $\tilde{\chi}'_1(X)$ and $\tilde{\chi}''_1(X)$ of §2. In addition, we are led to a concise formula (Theorem 4.10) for $\chi_1(X)(\gamma)$. We give two applications of these theorems to manifolds. Theorem 4.11 asserts that if X is a closed even dimensional smooth S^1 -manifold then $\chi_1(X)(\gamma) = 0$. Theorem 4.12 gives a formula for the Poincaré dual of the Euler class of the normal bundle to the flow defined by a smooth S^1 -action without fixed points on a smooth closed oriented manifold. Theorem 4.13 establishes the formulas (0.4) and (0.5) for 3-dimensional Seifert fibered spaces.

Let S^1 denote the set of complex numbers of unit modulus regarded as a compact Lie group. An S^1 -space is a topological space X together with a continuous left action, $S^1 \times X \xrightarrow{\alpha} X$, of S^1 on X .

We recall from [I₁] the notion of an “ S^1 -CW complex”.

Definition 4.1. Let X be a Hausdorff S^1 -space, A a closed S^1 -subset of X (i.e., $\alpha(S^1 \times A) \subset A$) and n a non-negative integer. We say X is obtained from A by attaching S^1 - n -cells if there is a collection $\{c_j^n \mid j \in J\}$ of closed S^1 -subsets of X such that:

- (1) $X = A \cup \bigcup_{j \in J} c_j^n$, and X has the topology coherent with A and $\{c_j^n \mid j \in J\}$.
- (2) For $i \neq j$, $(c_i^n - A) \cap (c_j^n - A) = \emptyset$.
- (3) For each $j \in J$ there is a closed subgroup H_j of S^1 (note that H_j is finite or $H_j = S^1$) and an S^1 -map $f_j: (S^1/H_j \times D^n, S^1/H_j \times S^{n-1}) \rightarrow (c_j^n, c_j^n \cap A)$ such that $f_j(S^1/H_j \times D^n) = c_j^n$ and f_j maps $S^1/H_j \times D^n - S^1/H_j \times S^{n-1}$ homeomorphically onto $c_j^n - A$.

Each c_j^n is called an S^1 - n -cell. The restriction $f_j|: S^1/H_j \times S^{n-1} \rightarrow A$ of f_j is called the *attaching map* of the S^1 - n -cell c^n .

Definition 4.2. An S^1 -relative CW complex (X, A) consists of a Hausdorff S^1 -space X , a closed S^1 -subset A of X and an increasing filtration of X by closed S^1 -subsets $(X, A)_k$, $k = 0, 1, \dots$, such that:

- (1) $(X, A)_0$ is obtained from A by attaching S^1 -0-cells, and for $k \geq 1$, $(X, A)_k$ is obtained from $(X, A)_{k-1}$ by attaching S^1 - k -cells.
- (2) $X = \bigcup_{k \geq 0} (X, A)_k$, and X has the topology coherent with $\{(X, A)_k \mid k \geq 0\}$.

The closed S^1 -subset $(X, A)_k$ is called the S^1 - k -skeleton of (X, A) . If $A = \emptyset$ we call X an S^1 -CW complex and denote its S^1 - k -skeleton by X_k . For convenience, we adopt the convention $X_{-1} = \emptyset$. We say X is *finite* if X has finitely many S^1 -cells.

Proposition 4.3. *Let X be a finite S^1 -CW complex. Then $(X, \{X_k\})$ is a filtered cell complex (Definition 3.1) in which the relative CW complex structure on (X_k, X_{k-1}) is related to the S^1 - k -cells c_j^k as follows: when H_j is finite, c_j^k is covered by one $(k+1)$ -cell d_j^{k+1} and one k -cell e_j^k of (X_k, X_{k-1}) ; when $H_j = S^1$, c_j^k is covered by one k -cell e_j^k . The images of the cells $\{e_j^k\}$ decompose the quotient space X/S^1 as a CW complex.*

Proof. Let Y be a space and $f: S^1 \times S^{k-1} \rightarrow Y$ a map and $f|: \{z_0\} \times S^{k-1} \rightarrow Y$ its restriction where $z_0 \in S^1$ is a basepoint. Let $Z = (S^1 \times D^k) \cup_f Y$ and $Z' = (\{z_0\} \times D^k) \cup_{f|} Y$ so that $Z' \subset Z$. Define $F: \partial(I \times D^k) \rightarrow Z'$ by $F(i, x) = (z_0, x)$ if $i = 0, 1$ and $x \in D^k$, and $F(t, y) = f(\exp(2\pi it), y)$ if $y \in S^{k-1}$ and $t \in I$. Then there is a natural homeomorphism $Z \cong (I \times D^k) \cup_F Z'$. Thus (Z, Y) is a relative CW complex, obtained by attaching a k -cell e and then a $(k+1)$ -cell d . Our claim for the case where H_j is finite follows. The case $H_j = S^1$ is similar. Although X is thus obtained by attaching cells, it might fail to be a CW complex only because the cells e_j^k might be attached to

the k -skeleton rather than the $(k-1)$ -skeleton; however, this difficulty disappears in the quotient space. \square

Theorem 7.1 and Proposition 6.1 of [I₂] give:

Proposition 4.4. *Any smooth S^1 -manifold M (with or without boundary) has an “ S^1 -triangulation” and thus an S^1 -CW complex structure (which is finite if M is compact) such that the attaching maps of the S^1 -cells are piecewise linear. In particular, an S^1 -triangulation gives rise to a polyhedral filtered cell complex (Definition 3.2). \square*

Let X be a connected finite S^1 -CW complex with base point x_0 , and let $G = \pi_1(X, x_0)$. The action $\alpha: S^1 \times X \rightarrow X$ is adjoint to a loop in X^X based at id_X , and hence defines a *fundamental element* $\gamma \in \Gamma_X$. Define $F^\gamma: X \times I \rightarrow X$ by $F^\gamma(x, t) = \alpha(e^{2\pi it}, x)$. Let the S^1 - n -cells be $\{c_j^n \mid j \in J_n\}$.

With each c_j^n we associate the *primitive loop* $\omega_j: S^1 \rightarrow X$: when H_j is finite, the formula is $\omega_j(e^{2\pi it}) = f_j(e^{2\pi it/|H_j|}, 0)$, an embedding of S^1 which goes around the S^1 -orbit of $\omega_j(1) \equiv f_j(1, 0)$ once. When $H_j = S^1$, ω_j is the constant loop at $f_j(S^1/S^1, 0)$. It is convenient to set $|H_j| = 0$ when $H_j = S^1$. Then the loops $\omega_j^{|H_j|}$ are freely homotopic for all n and all $j \in J_n$.

The choice of a lift \tilde{e}_j^n or \tilde{d}_j^{n+1} implies the choice of a base path σ_j (up to homotopy) for the cell e_j^n or d_j^{n+1} , with $\sigma_j(0) = x_0$. When H_j is finite, we pick abutting lifts \tilde{e}_j^n and \tilde{d}_j^{n+1} with the same base path σ_j ; more precisely (see the Sign Convention in §2) we arrange:

$$(4.5) \quad \tilde{\partial}_{n+1}^{n, n-1}(\tilde{d}_j^{n+1}) = (-1)^n \tilde{e}_j^n (g_{j,n}^{-1} - 1).$$

where $g_{j,n} \in G$ is represented by the loop $\sigma_j \omega_j \sigma_j^{-1}$. We have $g_{j,n}^{|H_j|} = \eta_\#(\gamma)$. In particular, if any $H_j = S^1$ then $\eta_\#(\gamma) = 1$.

Definition 4.6 (S^1 –Euler characteristic). $\tilde{\chi}_{S^1}(X) \in HH_1(\mathbb{Z}G)$ is the element represented by the Hochschild 1–cycle

$$\sum_{n \geq 0} (-1)^{n+1} \sum_{j \in J_n} \sum_{i=1}^{|H_j|} g_{j,n} \otimes g_{j,n}^{-1} g_{j,n}^{-i}.$$

Remark. $\tilde{\chi}_{S^1}(X)$ is independent of the choice of the basepaths σ_j . Another choice of basepaths gives rise to $g'_{j,n} \in G$ related to $g_{j,n}$ by $g'_{j,n} = h_{j,n} g_{j,n} h_{j,n}^{-1}$ for some $h_{j,n} \in G$. Then the corresponding cycles in Definition 4.6 are homologous because

$$g'_{j,n} \otimes (g'_{j,n})^{-1-i} - g_{j,n} \otimes g_{j,n}^{-1-i} = d(h_{j,n}^{-1} \otimes h_{j,n} g_{j,n} \otimes g_{j,n}^{-1-i} - h_{j,n} g_{j,n} \otimes h_{j,n}^{-1} \otimes (h_{j,n} g_{j,n} h_{j,n}^{-1})^{-1-i}).$$

By Proposition 4.3, we may regard X as a finite filtered cell complex (X, \mathcal{F}) , where $\mathcal{F} = \{X_n\}$ is the filtration by S^1 – n –skeleta. The cellular chain complex $C_*(\tilde{X}_n, \tilde{X}_{n-1})$ is zero in degrees other than n and $n+1$:

$$0 \rightarrow C_{n+1}(\tilde{X}_n, \tilde{X}_{n-1}) \xrightarrow{\tilde{\partial}_{n+1}^{n,n-1}} C_n(\tilde{X}_n, \tilde{X}_{n-1}) \rightarrow 0.$$

The chain homotopy $\tilde{D}_*^{\gamma,n,n-1}$ is given by

$$(4.7) \quad \tilde{D}_n^{\gamma,n,n-1}(\tilde{e}_j^n) = (-1)^{n+1} \tilde{d}_j^{n+1} \left(1 + g_{j,n}^{-1} + \cdots + g_{j,n}^{-(|H_j|-1)} \right)$$

(interpreted as 0 when $|H_j| = 0$). By our conventions, $\tilde{D}^{\gamma,n,n-1} = (-1)^{n+1} \tilde{D}_n^{\gamma,n,n-1}$ and $\tilde{\partial}^{n,n-1} = \tilde{\partial}_{n+1}^{n,n-1}$.

Proposition 4.8. $\tilde{\chi}_{S^1}(X) = \tilde{\mathcal{X}}_1(X; \mathcal{F})(\gamma)$

Proof. From Definition 3.3 and the expressions (4.5) and (4.7), we see that $\tilde{\mathcal{X}}_1(X; \mathcal{F})(\gamma)$ is represented by

$$\sum_{j \geq 0} (-1)^n (g_{j,n}^{-1} - 1) \otimes \left(\sum_{i=0}^{|H_j|-1} g_{j,n}^{-i} \right).$$

By standard Hochschild chain identities (see [GN₁; Lemma 6.13]), a term of the form $1 \otimes g$ is homologically trivial, and a term $g^{-1} \otimes g g^{-i}$ is homologous to $-g \otimes g^{-1} g^{-i}$. \square

Applying Theorem 3.6, to the right side of the equality $\tilde{\chi}_{S^1}(X) = \tilde{\mathcal{X}}_1(X; \mathcal{F})(\gamma)$ given by Proposition 4.8, we obtain.

Theorem 4.9. *If X is a connected finite S^1 -CW complex which is polyhedral as a filtered cell complex and $\gamma \in \Gamma_X$ is the fundamental element then:*

- (1) $\tilde{\mathcal{X}}_1''(X)(\gamma) = \pi''(\tilde{\chi}_{S^1}(X)).$
- (2) *Let $i: \langle \gamma \rangle \hookrightarrow \Gamma_X$ be the inclusion of the cyclic subgroup generated by γ . Then the cohomology class $i^* \tilde{\chi}_1'(X) \in H^1(\langle \gamma \rangle, HH_1(\mathbb{Z}G)')$ is represented by the derivation $d_X: \langle \gamma \rangle \rightarrow HH_1(\mathbb{Z}G)', d_X \equiv \tilde{\mathcal{X}}_1(X; \mathcal{F}) \circ i$, which takes γ to $\pi'(\tilde{\chi}_{S^1}(X)).$ \square*

Remark. Definition 4.6 has both a “combinatorial” and a geometric motivation. Note that $\tilde{\chi}_{S^1}(X)$ is defined in terms of the S^1 -CW structure of X (and so may be viewed as combinatorial); the defining formula is motivated by Proposition 4.8. Theorem 3.6 implies that $\tilde{\chi}_{S^1}(X)$ is essentially the same as $\tilde{\mathcal{X}}_1(X')(\gamma)$ where X' is a CW subdivision of X . If X' is a PL manifold then $\tilde{\mathcal{X}}_1(X')(\gamma)$ has a natural geometric interpretation in terms of fixed point theory and transversality as described in §2(C).

Notation. We write $\{g\} \in H_1(G) \equiv G_{ab}$ for $A(g)$, the image of $g \in G$ under abelianization.

The following theorem generalizes [GN₃; Theorem 4.5] where the case of a free S^1 -action is treated.

Theorem 4.10. *Let X be as in Theorem 4.9, If the S^1 -action has a fixed point then $\chi_1(X)(\gamma) = 0$. If there are no fixed points then $\chi_1(X)(\gamma) = -\chi(X/S^1)\{\eta_{\#}(\gamma)\}.$*

Proof. We have $|H_j|\{g_{j,n}\} = \{\eta_{\#}(\gamma)\}.$ By Propositions 1.1, 1.2 and 4.8,

$$\begin{aligned} \chi_1(X)(\gamma) &= \sum_{n \geq 0} (-1)^{n+1} \sum_{j \in J_n} |H_j|\{g_{j,n}\} \\ &= \sum_{n \geq 0} (-1)^{n+1} \sum_{j \in J_n} \{\eta_{\#}(\gamma)\} \end{aligned}$$

If there is a fixed point then $\eta_{\#}(\gamma) = 1$, so the right hand side is 0. If there is no fixed point, then the obvious bijection between the S^1 - n -cells of X and the n -cells of X/S^1 (Proposition 4.3) makes the right hand side $-\chi(X/S^1)\{\eta_{\#}(\gamma)\}.$ \square

Recall that there is a natural homomorphism $\varepsilon_*: HH_1(\mathbb{Z}G) \rightarrow H_1(G)$ which takes a Hochschild homology class represented by $\sum_i n_i g_i \otimes h_i$ to $\sum_i n_i \{g_i\}$ (see §1). The proof of Theorem 4.10 shows that:

Addendum. $\varepsilon_*(\tilde{\chi}_{S^1}(X)) = \chi_1(X)(\gamma)$.

We give two application to manifolds.

Our first application generalizes the classical theorem that the Euler characteristic of a closed odd dimensional manifold (whether orientable or not) is zero.

Theorem 4.11. *Let X be a closed even dimensional smooth manifold and suppose $\gamma \in \Gamma_X$ is represented by a smooth S^1 -action on X . Then $\chi_1(X)(\gamma) = 0$.*

Proof. By Proposition 4.4 there exists an S^1 -CW complex structure on the S^1 -manifold X . If the S^1 -action on X has a fixed point then by the first part of Theorem 4.10 we have $\chi_1(X)(\gamma) = 0$. If the S^1 -action has no fixed points then the quotient X/S^1 is a rational homology manifold of odd dimension and so $\chi(X/S^1) = 0$. Hence, by Theorem 4.10, $\chi_1(X)(\gamma) = -\chi(X/S^1)\{\eta_{\#}(\gamma)\} = 0$.

Remark. Theorem 4.11 generalizes [GNO, Corollary 7.5] where, using different techniques, the same conclusion (with rational coefficients) was obtained under the additional assumption that X is symplectic and the S^1 -action is Hamiltonian.

Our second application (Theorem 4.12) gives a formula for the Poincaré dual of the Euler class of the normal bundle to the flow defined by a smooth S^1 -action without fixed points on a closed oriented manifold. Let X be a connected smooth closed oriented S^1 -manifold partitioned as a finite S^1 -CW complex which is polyhedral as a filtered cell complex (see Proposition 4.4). Assume that the action has no fixed points, so that each H_j is finite. Let λ be the real line bundle over X consisting of tangent vectors which are tangent to the S^1 -orbits. The S^1 -action determines an orientation on λ . Let $\nu = T_X/\lambda$ where T_X is the tangent bundle of X . Given a Riemannian metric on X , ν is identified with the

oriented normal bundle to the flow defined by the S^1 -action. Recall from [GN₃] that (in view of our use of Dold's sign conventions for cap products) Poincaré duality is given by $\text{PD}_X(u) = (-1)^{i(m-i)}u \cap [X]$ where $u \in H^i(X)$ and $[X] \in H_m(X)$ is the fundamental class of X . We compute the Poincaré dual of the Euler class $\text{Eul}(\nu) \in H^{m-1}(X; \mathbb{Z})$:

Theorem 4.12. $\text{PD}_X(\text{Eul}(\nu)) = \sum_{n=0}^{m-1} (-1)^n \sum_{j \in J_n} \{g_{j,n}\} \in H_1(X; \mathbb{Z})$.

Proof. Using Definition C₁ of $\chi_1(X)(\gamma)$ given in [GN₃], and applying Theorems 2.5 and 3.1 of [GN₂], we get

$$\chi_1(X)(\gamma) = \left(\sum_{n=0}^{m-1} (-1)^{n+1} \sum_{j \in J_n} (|H_j| - 1) \{g_{j,n}\} \right) - \text{PD}_X(\text{Eul}(\nu)).$$

Comparing this with the formula in the proof of Theorem 4.10 yields the claimed result.

□

Remarks.

- (1) Here is an informal explanation of the proof of Theorem 4.12. Assume that $F^\gamma: X \times I \rightarrow X$ has no fixed points in $X \times (0, \epsilon]$ where $\epsilon > 0$. Moving the $[1 - \epsilon, 1]$ portion to $[-\epsilon, 0]$, let $\bar{F}^\gamma: X \times [-\epsilon, 1 - \epsilon] \rightarrow X$ be the reorganized homotopy. The contribution of $\bar{F}^\gamma|_{X \times [-\epsilon, \epsilon]}$ to $\chi_1(X)(\gamma)$ is $-\text{PD}_X(\text{Eul}(\nu))$; this is Theorem 3.1 of [GN₂]. The contribution of $\bar{F}^\gamma|_{X \times [\epsilon, 1 - \epsilon]}$ to $\chi_1(X)(\gamma)$ is the summation term; this is Theorem 2.5 of [GN₂].
- (2) In Theorem 4.12 the right side is clearly independent of the choice of orientation for X ; however, so is the left side because $\text{Eul}(-\nu) \cap [-X] = \text{Eul}(\nu) \cap [X]$.

Anticipating §6, we apply Theorem 4.12 in the case X is a compact connected 3-dimensional Seifert fibered space. We will call a Seifert fibered space X *admissible* if X is oriented, the quotient surface Σ is oriented, and X is not one of the special cases: $\Sigma = S^2$ with one or two exceptional fibers, or $\Sigma = D^2$ with one exceptional fiber. Theorem 4.12 and Theorem 6.2(1) yield the following formula:

Theorem 4.13. *Let X be an admissible 3-dimensional Seifert fibered space with quotient surface Σ and r exceptional fibers. Impose the compatible orientation on the fibers and on the normal bundle ν (to the Seifert fibering). Let $\{\gamma_0\} \in H_1(X; \mathbb{Z})$ be the homology class of an ordinary fiber, and let $\{g_j\}$ be the homology class of the j -th singular fiber traversed once in the positive direction. Then the Poincaré dual of the Euler class of the oriented bundle ν is*

$$\text{PD}(\text{Eul}(\nu)) = (\chi(\Sigma) - r)\{\gamma_0\} + \sum_{j=1}^r \{g_j\} \in H_1(X; \mathbb{Z}). \quad \square$$

§5. T^2 -ACTIONS

In this section we show that our invariants vanish for S^1 -actions which extend to T^2 -actions with finite isotropy. The main results are Theorem 5.3, Theorem 5.4 and Corollary 5.5.

Let $T^2 \equiv S^1 \times S^1$ denote the 2-torus viewed as a compact Lie group. A T^2 -space is a topological space X together with a continuous left T^2 -action $T^2 \times X \rightarrow X$. Given any circle subgroup $S^1 \subset T^2$, the restriction of $T^2 \times X \rightarrow X$ to $S^1 \times X \rightarrow X$ yields a circle action. Define $\Gamma_X^{T^2} \subset \Gamma_X$ to be the subgroup generated by the fundamental elements (see §4) associated with circle actions obtained in this manner. Note that $\Gamma_X^{T^2}$ is the image of the homomorphism $\pi_1(T^2, 1) \rightarrow \Gamma_X$ induced by the map $T^2 \rightarrow X^X$ which is adjoint to the action map $T^2 \times X \rightarrow X$.

Let X be a T^2 -CW complex (the notion of a K -CW complex, for any compact Lie group K is defined in [I₁]; see Definitions 4.1 and 4.2 for the case $K = S^1$). For each T^2 - n -cell, $c_j^n \subset X$, there is a closed subgroup $H_j \subset T^2$ and a T^2 -map

$$f_j: (T^2/H_j \times D^n, T^2/H_j \times S^{n-1}) \rightarrow (c_j^n, \partial c_j^n)$$

such that $f_j(T^2/H_j \times D^n) = c_j^n$ and f_j maps $T^2/H_j \times D^n - T^2/H_j \times S^{n-1}$ homeomorphically onto $c_j^n - \partial c_j^n$. The subgroup H_j is called the *isotropy group* of c_j^n and the restriction

of f_j to $T^2/H_j \times S^{n-1}$ is called the *attaching map* of c_j^n . Let $\mathcal{F} = \{X_n\}$ be the filtration of X by the T^2 - n -skeleta (X_n is the union of the T^2 - k -cells of X , $k \leq n$). We describe a cell structure on $\tilde{T}^2 \equiv \mathbb{R}^2$ which will be used to give (X, \mathcal{F}) the structure of a filtered cell complex. Let $p: \mathbb{R}^2 \rightarrow T^2$ be the universal covering projection, $p(t_1, t_2) = (e^{2\pi i t_1}, e^{2\pi i t_2})$. The group of covering translations is \mathbb{Z}^2 acting by $(m, n) \cdot (t_1, t_2) = (t_1 + m, t_2 + n)$ where $(m, n) \in \mathbb{Z}^2$ and $(t_1, t_2) \in \mathbb{R}^2$. Let $I = [0, 1]$. The cells $\tilde{E}^2 = I \times I$, $\tilde{E}_1^1 = I \times \{0\}$, $\tilde{E}_2^1 = \{0\} \times I$, $\tilde{E}^0 = \{(0, 0)\}$ together with their translates give \mathbb{R}^2 a CW structure; their images $E^2 = p(\tilde{E}^2)$, $E_1^1 = p(\tilde{E}_1^1)$, $E_2^1 = p(\tilde{E}_2^1)$, $E^0 = p(\tilde{E}^0)$ give T^2 a CW structure. Let (D^n, S^{n-1}) have the standard relative CW structure (with a single n -cell). The space T^2/H_j is given a CW structure which depends on the dimension of the subgroup H_j :

- (1) If $\dim H_j = 2$, $H_j = T^2$ and so T^2/H_j is a single point.
- (2) If $\dim H_j = 1$, choose an orientation preserving isomorphism of Lie groups $S^1 \cong T^2/H_j$. Give S^1 the CW structure consisting of one 0-cell and one 1-cell and give T^2/H_j the CW structure induced by the chosen isomorphism.
- (3) If $\dim H_j = 0$, choose an orientation preserving isomorphism of Lie groups $T^2 \cong T^2/H_j$. Give T^2 the CW structure described above and give T^2/H_j the CW structure induced by the chosen isomorphism.

The pair $(T^2/H_j \times D^n, T^2/H_j \times S^{n-1})$ is given the product relative CW structure. The maps f_j determine a relative CW structure on (X_n, X_{n-1}) thus realizing (X, \mathcal{F}) as a filtered cell complex.

By Theorem 7.1 and Proposition 6.1 of [I₂] (where the case of a general compact Lie group action is treated), we have:

Proposition 5.1. *Any smooth T^2 -manifold M (with or without boundary) has a “ T^2 -triangulation” and thus a T^2 -CW complex structure (which is finite if M is compact) such that the attaching maps of the T^2 -cells are piecewise linear. In particular, a T^2 -triangulation gives rise to a polyhedral filtered cell complex.* □

Given a pair of integers (a, b) , define $\tilde{F}^{(a,b)}: \mathbb{R}^2 \times I \rightarrow \mathbb{R}^2$ by

$$\tilde{F}^{(a,b)}((t_1, t_2), t) = (t_1 + ta, t_2 + tb) \quad (t_1, t_2) \in \mathbb{R}^2, t \in I.$$

Note that $\tilde{F}^{(a,b)}$ descends to a map $F^{(a,b)}: T^2 \times I \rightarrow T^2$ and any map $F: T^2 \times I \rightarrow T^2$ with $F_0 = F_1 = \text{id}$ is homotopic rel $T^2 \times \{0, 1\}$ to $F^{(a,b)}$ for some unique (a, b) . This establishes an isomorphism of groups $\Gamma_{T^2} \xrightarrow{\cong} \mathbb{Z} \times \mathbb{Z}$ where $\gamma \in \Gamma_{T^2}$, represented by $F^{(a,b)}: T^2 \times I \rightarrow T^2$, is mapped to (a, b) . The map $F^{(a,b)}$ is not necessarily cellular with respect to the cell structure we have imposed on T^2 ; however, $F^{(a,b)}$ is homotopic rel $T^2 \times \{0, 1\}$ to the cellular map $G^{(a,b)}: T^2 \times I \rightarrow T^2$ whose lift $\tilde{G}^{(a,b)}: \mathbb{R}^2 \times I \rightarrow \mathbb{R}^2$ is given by

$$\tilde{G}^{(a,b)}((t_1, t_2), t) = \begin{cases} (t_1, t_2 + 2tb) & \text{if } 0 \leq t \leq \frac{1}{2} \\ (t_1 + (2t - 1)a, t_2 + b) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Let $H \equiv \pi_1(T^2, (1, 1)) \cong \mathbb{Z} \times \mathbb{Z}$. The *standard generators* of H are the elements x_1, x_2 represented respectively by $I \rightarrow S^1 \times S^1, t \mapsto (e^{2\pi it}, 1)$ and $I \rightarrow S^1 \times S^1, t \mapsto (1, e^{2\pi it})$. For any $x \in H$ and integer m define $x^{[m]} \in \mathbb{Z}H$ by

$$x^{[m]} = \begin{cases} (1 + x + \cdots + x^{m-1}) & \text{if } m > 0 \\ 0 & \text{if } m = 0 \\ (-x^{-1} - x^{-2} - \cdots - x^{-m}) & \text{if } m < 0. \end{cases}$$

Then it is straightforward to show:

Lemma 5.2. *The chain homotopy $\tilde{D}_*^{(a,b)}: C_*(\tilde{T}^2) \rightarrow C_{*+1}(\tilde{T}^2)$ associated to $G^{(a,b)}$ is given by*

$$\begin{aligned} \tilde{D}_0^{(a,b)}(\tilde{E}^0) &= \tilde{E}_1^1(x_1^{-1})^{[a]}x_2^{-b} + \tilde{E}_2^1(x_2^{-1})^{[b]} \\ \tilde{D}_1^{(a,b)}(\tilde{E}_1^1) &= -\tilde{E}^2(x_2^{-1})^{[b]} \\ \tilde{D}_1^{(a,b)}(\tilde{E}_2^1) &= \tilde{E}^2(x_1^{-1})^{[a]}x_2^{-b}. \quad \square \end{aligned}$$

Theorem 5.3. *Let X be a finite connected T^2 -CW complex such that all the isotropy groups H_j are finite. Then $\tilde{\mathcal{X}}_1(X; \mathcal{F})(\gamma) = 0$ for all $\gamma \in \Gamma_X^{T^2}$.*

Proof. For each T^2 - n -cell c_j^n with associated map $f_j: T^2/H_j \times D^n \rightarrow c_j^n$ define

$$g_{j,i,n} \equiv (f_j \circ (h, k))_{\#}(x_i) \in G \equiv \pi_1(X, v), \quad i = 1, 2$$

where $k: T^2 \rightarrow D^n$ is the constant map (taking T^2 to the basepoint of D^n), $h: T^2 \xrightarrow{\cong} T^2/H_j$ is the isomorphism used to endow T^2/H_j with its CW structure (note that H_j is finite by hypothesis) and x_i , $i = 1, 2$, are the standard generators of $\pi_1(T^2, (1, 1))$. Let e_j^{n+2} , $e_{j,i}^{n+1}$, $i = 1, 2$, and e_j^n be the cells given by $e_j^{n+2} = f_j \circ (h \times \text{id})(E^2 \times D^n)$, $e_{j,i}^{n+1} = f_j \circ (h \times \text{id})(E_i^1 \times D^n)$, $i = 1, 2$, and $e_j^n = f_j \circ (h \times \text{id})(E^0 \times D^n)$ and let \tilde{e}_j^{n+2} , $\tilde{e}_{j,i}^{n+1}$, $i = 1, 2$, and \tilde{e}_j^n be corresponding lifts of these cells to \tilde{X} . The relative cellular chain complex $C_*(\tilde{X}_n, \tilde{X}_{n-1})$ is zero in degrees other than n , $n+1$, $n+2$:

$$0 \rightarrow C_{n+2}(\tilde{X}_n, \tilde{X}_{n-1}) \xrightarrow{\tilde{\partial}_{n+2}^{n,n-1}} C_{n+1}(\tilde{X}_n, \tilde{X}_{n-1}) \xrightarrow{\tilde{\partial}_{n+1}^{n,n-1}} C_n(\tilde{X}_n, \tilde{X}_{n-1}) \rightarrow 0$$

where the boundary operators are given by

$$\begin{aligned} \tilde{\partial}_{n+2}^{n,n-1}(\tilde{e}_j^{n+2}) &= (-1)^n (\tilde{e}_{j,1}^{n+1}(1 - g_{j,2,n}^{-1}) - \tilde{e}_{j,2}^{n+1}(1 - g_{j,1,n}^{-1})) \\ \tilde{\partial}_{n+1}^{n,n-1}(\tilde{e}_{j,1}^{n+1}) &= (-1)^{n+1} \tilde{e}_j^n (1 - g_{j,1,n}^{-1}) \\ \tilde{\partial}_{n+1}^{n,n-1}(\tilde{e}_{j,2}^{n+1}) &= (-1)^{n+1} \tilde{e}_j^n (1 - g_{j,2,n}^{-1}). \end{aligned}$$

Let $\gamma \in \Gamma_X^{T^2}$ be the fundamental element corresponding to a circle subgroup $j: S^1 \hookrightarrow T^2$, i.e., if $\alpha: T^2 \times X \rightarrow X$ is the action map then γ is represented by $F^\gamma: X \times I \rightarrow X$, $F^\gamma(x, t) = \alpha(j(e^{2\pi it}), x)$. Clearly, F^γ preserves the filtration \mathcal{F} ; however, the maps $F^{\gamma,n,n-1}: (X_n, X_{n-1}) \times I \rightarrow (X_n, X_{n-1})$ are not necessarily cellular. Using Cellular Approximation, Lemma 5.2 and the discussion preceding it, we can replace

F^γ by a filtration preserving map G^γ such that the maps $G^{\gamma,n,n-1}$ are cellular and the corresponding chain homotopies, $\tilde{D}_*^{\gamma,n,n-1}$, are given by:

$$\begin{aligned}\tilde{D}_n^{\gamma,n,n-1}(\tilde{e}_j^n) &= \tilde{e}_{j,1}^{n+1}(g_{j,1,n}^{-1})^{[a_j]}g_{j,2,n}^{-b_j} + \tilde{e}_{j,2}^{n+1}(g_{j,2,n}^{-1})^{[b_j]} \\ \tilde{D}_{n+1}^{\gamma,n,n-1}(\tilde{e}_{j,1}^{n+1}) &= -\tilde{e}_j^{n+2}(g_{j,2,n}^{-1})^{[b_j]} \\ \tilde{D}_{n+1}^{\gamma,n,n-1}(\tilde{e}_{j,2}^{n+1}) &= \tilde{e}_j^{n+2}(g_{j,1,n}^{-1})^{[a_j]}g_{j,2,n}^{-b_j}\end{aligned}$$

where (a_j, b_j) is a pair of integers determined by the circle subgroup $j: S^1 \hookrightarrow T^2$ and the isotropy group H_j . We write $\tilde{\partial}^{n,n-1} = \bigoplus_k \tilde{\partial}_k^{n,n-1}$, $\tilde{D}^{\gamma,n,n-1} = \bigoplus_k (-1)^{k+1} \tilde{D}_k^{\gamma,n,n-1}$ (viewed as matrices over $\mathbb{Z}G$; see §2.). Then

$$\begin{aligned}\text{trace}(\tilde{\partial}^{n,n-1} \otimes \tilde{D}^{\gamma,n,n-1}) &= \sum_j \left((-1)^n (1 - g_{j,2,n}^{-1}) \otimes -(-1)^{n+1} (g_{j,2,n}^{-1})^{[b_j]} \right. \\ &\quad + (-1)^n (1 - g_{j,1,n}^{-1}) \otimes (-1)^{n+1} (g_{j,1,n}^{-1})^{[a_j]} g_{j,2,n}^{-b_j} \\ &\quad + (-1)^{n+1} (1 - g_{j,1,n}^{-1}) \otimes (-1)^n (g_{j,1,n}^{-1})^{[a_j]} g_{j,2,n}^{-b_j} \\ &\quad \left. + (-1)^{n+1} (1 - g_{j,2,n}^{-1}) \otimes (-1)^n (g_{j,2,n}^{-1})^{[b_j]} \right).\end{aligned}$$

By an obvious cancellation of terms with opposite sign on the right side of the above expression, we conclude $\text{trace}(\tilde{\partial}^{n,n-1} \otimes \tilde{D}^{\gamma,n,n-1}) = 0$. Hence

$$\tilde{\mathcal{X}}_1(X; \mathcal{F})(\gamma) = \sum_n T_1(\tilde{\partial}^{n,n-1} \otimes \tilde{D}^{\gamma,n,n-1}) = \sum_n 0 = 0. \quad \square$$

Applying Theorem 3.6, we deduce:

Theorem 5.4. *Suppose X is a connected finite T^2 -CW complex such that all the isotropy groups are finite and that X is polyhedral as a filtered cell complex. Let $i: \Gamma_X^{T^2} \rightarrow \Gamma_X$ be the inclusion. Then $\tilde{\mathcal{X}}_1''(X) \circ i = 0$ and $i^* \tilde{\mathcal{X}}_1'(X) = 0$. \square*

By Theorem 4.9, we obtain the following “vanishing theorem” for the S^1 -Euler characteristic of §4 in the presence of a T^2 -action with finite isotropy:

Corollary 5.5. *Suppose X is a connected finite S^1 -CW complex which is polyhedral as a filtered cell complex and there exists a T^2 -action on X with finite isotropy groups such that X has a T^2 -CW complex structure and the given S^1 -action is obtained by restriction of the T^2 -action to a circle subgroup. Then $\pi''(\tilde{\chi}_{S^1}(X)) = 0$ and the derivation $\langle \gamma \rangle \rightarrow HH_1(\mathbb{Z}G)'$ which takes γ to $\pi'(\tilde{\chi}_{S^1}(X))$ is inner where $\gamma \in \Gamma_X$ is the fundamental element of the given S^1 -action. \square*

In particular, the S^1 -Euler characteristic can be viewed as an obstruction to the existence of a T^2 -action with finite isotropy.

§6. COMPUTATIONS FOR 3-DIMENSIONAL SEIFERT FIBERED SPACES

In this section we compute $\tilde{\chi}_{S^1}(X)$ when X is an oriented 3-dimensional Seifert fibered space (Theorem 6.2) and discuss how much of that structure is detected by $\tilde{\chi}_{S^1}(X)$ (Theorem 6.7).

Consider a smooth S^1 -action without fixed points on a compact connected oriented 3-manifold X . The orbits of such an action give X the structure of a Seifert fibered space whose fibers are consistently oriented, and, conversely, any such Seifert fibered space comes from an S^1 -action (see [Sc, p.430]). A Seifert fibering has only finitely many singular fibers. By Proposition 4.4, the S^1 -space X can be given the structure of an S^1 -CW complex which is polyhedral as a filtered cell complex so that the singular fibers are S^1 -0-cells. Thus there are only finitely many subgroups of S^1 which are orbit stabilizers. Factoring out the intersection of these stabilizers if necessary, we will assume that the S^1 -action is free away from the singular fibers.³

Our trace invariants are features of the S^1 -action on X , but a Seifert fibering, even when the fibers can be consistently oriented, is a different type of structure on X . In this

³General references for the material on Seifert fibered spaces used here are [B], [J], [JS] and [Se].

section we explore some of the ways in which our invariants give information about the Seifert fibering.

The stabilizer H_j is trivial unless c_j^0 is a singular fiber, in which case we write $\mu_j \equiv |H_j| > 1$. Let $J'_0 \subset J_0$ be the indexing set for the singular fibers (as in §4, J_0 indexes the S^1 -0-cells). We write $r \equiv |J'_0|$, the number of singular fibers, and, abusing notation, we sometimes identify J'_0 with $\{1, \dots, r\}$. We assume the basepoint x_0 lies in an ordinary fiber. When making use of the homomorphism $\eta_\# : \Gamma_X \rightarrow G \equiv \pi_1(X, x_0)$, we write $\gamma_0 = \eta_\#(\gamma)$, $\alpha_0 = \eta_\#(\alpha)$, etc., (recall that $\Gamma_X = \pi_1(X^X, \text{id}_X)$ where X^X is the function space of self-maps of X and that $\eta_\#$ is induced by evaluation at the basepoint). For $j \in J'_0$ we abbreviate $g_{j,0}$ to g_j . As in §4, the S^1 -action defines $F^\gamma : X \times I \rightarrow X$ where $\gamma \in \Gamma_X$ is the fundamental element. Note that $g_j^{\mu_j} = \gamma_0 \in Z(G)$ for each $j \in J'_0$ (recall that $Z(G)$ denotes the center of G).

The quotient space $\Sigma \equiv X/S^1$ is a compact connected surface. A consistent orientation on the fibers of X imposes an orientation on Σ . Conversely, if Σ is orientable then the fibers of X can be consistently oriented.

We continue to assume X and Σ are oriented, and we recall some facts about the fundamental group of X .

Case 1. $\partial X = \emptyset$. The Seifert fibering is completely determined by Σ , pairs of integers (μ_j, β_j) for $1 \leq j \leq r$, and an integer b . Here, the positive integers μ_j are as above. For each singular fiber c_j^0 , there is a number $0 < \nu_j < \mu_j$ with $\gcd(\nu_j, \mu_j) = 1$ so that a fibered solid torus neighborhood of c_j^0 is obtained from a standard fibered solid torus by cutting and regluing using a $2\pi\nu_j/\mu_j$ twist; and integers α_j, β_j are chosen so that $\alpha_j\mu_j + \beta_j\nu_j = 1$ and $0 < \beta_j < \mu_j$. The integer b measures the obstruction to constructing a section $\Sigma \rightarrow X_0$ where X_0 is obtained from X by drilling out the singular fibers and filling the holes with standard fibered solid tori.

Writing σ for the genus of Σ , a well known presentation of G (see [B]) is:

$$\langle \gamma_0, a_1, \dots, a_\sigma, b_1, \dots, b_\sigma, c_1, \dots, c_r \mid \gamma_0 \text{ is central, } c_j^{\mu_j} \gamma_0^{\beta_j} = 1, \prod_{i=1}^{\sigma} [a_i, b_i] \prod_{j=1}^r c_j \gamma_0^b = 1 \rangle$$

This presentation is equivalent by Tietze transformations $g_j \equiv c_j^{-\nu_j} \gamma_0^{\alpha_j}$ to the presentation:

$$\langle \gamma_0, a_1, \dots, a_\sigma, b_1, \dots, b_\sigma, g_1, \dots, g_r \mid \gamma_0 \text{ is central, } g_j^{\mu_j} = \gamma_0, \prod_{i=1}^{\sigma} [a_i, b_i] \prod_{j=1}^r g_j^{-\beta_j} \gamma_0^b = 1 \rangle$$

One thinks of the a_i 's and b_i 's as generators of $\pi_1(\Sigma)$ and of γ_0 and g_j as having their previous meanings.

Case 2. $\partial X \neq \emptyset$. Then X is aspherical and G is infinite. We saw in §2 that in the aspherical case $\Gamma_X \cong Z(G)$ so our trace invariants only depend on the fundamental group.⁴ In this case, after Tietze transformations as above, G has a presentation:

$$\langle \gamma_0, a_1, \dots, a_\sigma, b_1, \dots, b_\sigma, g_1, \dots, g_r, d_1, \dots, d_{m-1} \mid \gamma_0 \text{ is central, } g_j^{\mu_j} = \gamma_0 \rangle$$

Here, $m > 0$ is the number of boundary components of Σ , while σ , g_j , γ_0 and μ_j are as before. The numbers β_j are absent from the presentation because we “solved” for d_m and removed it from the set of generators as we removed the only relation involving the β_j 's.

We will call a Seifert fibered space X *admissible* if X is oriented, Σ is oriented, and X is not one of the special cases: $\Sigma = S^2$ and $r = 1$ or 2 , or $\Sigma = D^2$ and $r = 1$.

Recall that the Gottlieb subgroup of G , denoted by $\mathcal{G}(X)$, is the image of the homomorphism $\eta_\# : \Gamma_X \rightarrow G$ induced by evaluation at the basepoint. Our computations will depend on the following basic fact:

⁴While in the closed aspherical manifold case the fundamental group determines the Seifert fibered space up to homeomorphism, this is not so when there is a non-empty boundary, e.g. if Σ_1 is a surface of genus 0 with three boundary components and Σ_2 is a surface of genus 1 with one boundary component then the aspherical Seifert fibered manifolds $S^1 \times \Sigma_1$ and $S^1 \times \Sigma_2$ have the same fundamental group but are not homeomorphic.

Proposition 6.1. *Let X be admissible. If $0 < i < \mu_j$ then $g_j^i \notin Z(G)$, hence $g_j^i \notin \mathcal{G}(X)$. If $0 < k < \mu_\ell$ and $(j, i) \neq (\ell, k)$ then g_j^i is not conjugate to g_ℓ^k .*

Remark. This is false when X is not admissible.

Proof of 6.1. This is well known to experts so we confine ourselves to a short sketch.

Case 1: $\partial X \neq \emptyset$; then the claimed facts can be read off from the given presentation of G by killing γ_0 and each a_i , b_i and d_i , except for the non-admissible case $\Sigma = D^2$ and $r = 1$.

Case 2: $\partial X = \emptyset$ and $r \geq 3$; then one finds an appropriate triangle group as Fuchsian quotient; the corresponding claims in the triangle group follow from the geometry of its well known action on hyperbolic or euclidean 2-space or on the 2-sphere (see [Si]).

Case 3: $\partial X = \emptyset$ and $r \leq 2$; if $r = 0$ there is nothing to prove, so assume $r = 1$ or 2 . If Σ has genus ≥ 2 , split Σ along a circle so that each part has genus at least 1, and arrange X so that the singular fibers do not lie over this circle, and, if there are two, that they lie one over each side. This splits G as a free product with amalgamation, and a simple geometric argument establishes the claim. A similar HNN argument handles the case where Σ has genus 1. \square

We saw in §1 that $HH_*(\mathbb{Z}G) \cong \bigoplus_{C \in G_1} H_*(Z(g_C))$ by an isomorphism which is canonical once a representative g_C has been chosen for each $C \in G_1$ (recall that $Z(g)$ denotes the centralizer of $g \in G$ and that G_1 is the set of conjugacy classes of G). In the present case we choose γ_0^{-1} and g_j^{-i} as representatives of their conjugacy classes, $j \in J'_0$, $0 < i < \mu_j$. Since $\gamma_0 \in Z(G)$, we have $G = Z(\gamma_0)$ and so $H_1(G) \equiv H_1(Z(\gamma_0)) \cong HH_1(\mathbb{Z}G)_{C(\gamma_0)}$. We remind the reader of the following notation convention: given a subgroup $K \subset G$ and $g \in K$ we write $\{g\} \in H_1(K) \equiv K_{\text{ab}}$ for the image of g under abelianization $K \rightarrow K_{\text{ab}}$.

Theorem 6.2. *Let X be admissible. The components of $\tilde{\chi}_{S^1}(X)$ in*

$\bigoplus_{C \in G_1} H_1(Z(g_C)) \cong HH_1(\mathbb{Z}G)$ are:

- (1) *The $C(\gamma_0)$ -component is $(r - \chi(\Sigma))\{\gamma_0\} - \sum_{j=1}^r \{g_j\} \in H_1(G)$.*
- (2) *The $C(g_j^i)$ -component is $-\{g_j\} \in H_1(Z(g_j^i))$ for each $j \in J'_0$ and $0 < i < \mu_j$.*
- (3) *All other components are zero.* □

Remark. In the notation of §2(A), (1) is the $HH_1(\mathbb{Z}G)'$ part and (2) is the $HH_1(\mathbb{Z}G)''$ part.

Proof. In canonical form, $\tilde{\chi}_{S^1}(X)$ is represented by the cycle

$$\zeta = \sum_{n \geq 0} (-1)^{n+1} \sum_{j \in J_n - J'_0} \gamma_0 \otimes \gamma_0^{-1} \gamma_0^{-1} - \sum_{j=1}^r \sum_{i=1}^{\mu_j} g_j \otimes g_j^{-1} g_j^{-i} = \zeta' + \zeta''$$

where

$$\zeta' = \sum_{n \geq 0} (-1)^{n+1} \sum_{j \in J_n - J'_0} \gamma_0 \otimes \gamma_0^{-1} \gamma_0^{-1} - \sum_{j \in J'_0} g_j \otimes g_j^{-1} \gamma_0^{-1}$$

and

$$\zeta'' = - \sum_{j=1}^r \sum_{i=1}^{\mu_j-1} g_j \otimes g_j^{-1} g_j^{-i}.$$

By Proposition 6.1, this decomposes ζ into a single central component and various components whose markers are not in $\mathcal{G}(X)$.

For any $a \in G$, $a \otimes a^{-2}$ is homologous to $-a^{-1} \otimes 1 \equiv -a^{-1} \otimes aa^{-1}$. Thus, writing “ $u \sim v$ ” when u and v are homologous chains in the Hochschild complex (see §1), we have

$$\zeta' \sim (\chi(\Sigma) - r) \gamma_0^{-1} \otimes \gamma_0 \gamma_0^{-1} + \sum_{j=1}^r g_j^{-1} \otimes g_j \gamma_0^{-1}$$

and

$$\zeta'' \sim \theta'' \equiv \sum_{j=1}^r \sum_{i=1}^{\mu_j-1} g_j^{-1} \otimes g_j g_j^{-i}.$$

For $c \in Z(G)$ and $a, b \in G$, we have (see [GN₁, Lemma 6.13])

$$ab^{-1} \otimes ba^{-1}c \sim a \otimes a^{-1}c - b \otimes b^{-1}c.$$

So $\zeta' \sim \theta' \equiv h \otimes h^{-1} \gamma_0^{-1}$ where $h = \gamma_0^{r-\chi(\Sigma)} g_1^{-1} \cdots g_r^{-1}$. Thus $\theta \equiv \theta' + \theta''$ is a representative Hochschild cycle for $\tilde{\chi}_{S^1}(X)$ in which every term lies in a different component $C_*(\mathbb{Z}G)_C$. Collecting terms,

$$\theta \equiv \theta' + \theta'' = h \otimes h^{-1} \gamma_0^{-1} + \sum_{j=1}^r \sum_{i=1}^{\mu_j-1} g_j^{-1} \otimes g_j g_j^{-i}.$$

where $h = \gamma_0^{r-\chi(\Sigma)} g_1^{-1} \cdots g_r^{-1}$. The conclusion of the theorem follows from Proposition 1.1 applied to this expression for θ . \square

By Theorem 4.9(2), Theorem 6.2(1) and Theorem A.2 we have:

Proposition 6.3. *Let X be admissible and suppose $(\chi(\Sigma) - r)\{\gamma_0\} + \sum_{j=1}^r \{g_j\} \neq 0 \in H_1(X; \mathbb{Z})$. Then the Gottlieb subgroup $\mathcal{G}(X)$ is cyclic and is generated by γ_0 .* \square

Corollary 6.4. *If in addition X is aspherical then $Z(G)$ is infinite cyclic and is generated by γ_0 .*

Proof. By a well known argument (see [CR, p.43]) γ_0 is non-trivial when X is aspherical. \square

When $\{\gamma_0\}$ is non-zero in $H_1(G)$ each $\{g_j\}$ is non-zero in $H_1(Z(g_j^i))$ for $0 < i < \mu_j$ since $g_j^{\mu_j} = \gamma_0$. Thus, combining Theorems 4.9 and 4.10 and Theorem 6.2, we see what features of the Seifert fibering are detected by the *topological* invariants $\tilde{\chi}_1''(X)$, $\tilde{\chi}_1'(X)$ and $\chi_1(X)$ of §2:

Proposition 6.5. *If $\{\gamma_0\} \neq 0 \in H_1(G)$ and X is admissible then:*

- (1) $\tilde{\chi}_1''(X)(\gamma)$ detects the number, r , of singular fibers, the numbers μ_j and the conjugacy classes of the singular fibers.
- (2) $\chi_1'(X)(\gamma) = (r - \chi(\Sigma))\{\gamma_0\} - \sum_{j=1}^r \{g_j\} \in H_1(G)$.
- (3) $\chi_1(X)(\gamma) = -\chi(\Sigma)\{\gamma_0\} \in H_1(G)$. \square

Next, we mention the connection with two classical invariants of Seifert fibered spaces, the Euler number and the orbifold Euler characteristic. When $\partial X = \emptyset$ the *Euler number* of the Seifert fibered space X (see [Sc, §5]) is defined to be the rational number $b - \sum_{j=1}^r \beta_j / \mu_j$. By abelianizing the given presentations for G we conclude a well known fact:

Lemma 6.6. *$\{\gamma_0\} \in H_1(G)$ has infinite order if and only if either $\partial X \neq \emptyset$ or $\partial X = \emptyset$ and the Euler number of X is zero.* \square

Combining Lemma 6.6 with Theorems 6.2 and 4.10 we obtain:

Theorem 6.7. *Suppose that X is admissible and either $\partial X \neq \emptyset$ or the Euler number is zero. Then $\tilde{\chi}_{S^1}(X)$ determines the following features of the Seifert fibering: $\chi(\Sigma)$, the number of singular fibers r , the integers μ_1, \dots, μ_r , and the conjugacy classes in G represented by the singular fibers.* \square

The *orbifold Euler characteristic* of the quotient surface Σ (viewed as an orbifold) is the rational number $\chi_V(\Sigma) \equiv \chi(\Sigma) + \sum_{j=1}^r (\frac{1}{\mu_j} - 1)$. Denote the image of a homology class $z \in H_1(X; \mathbb{Z})$ in $H_1(X; \mathbb{Q})$ by $z_{\mathbb{Q}}$.

Proposition 6.8. $\pi'(\tilde{\chi}_{S^1}(X))_{\mathbb{Q}} = -\chi_V(X)\{\gamma_0\}_{\mathbb{Q}}$.

Proof. We have $\{g_j\}_{\mathbb{Q}} = \frac{1}{\mu_j}\{\gamma_0\}_{\mathbb{Q}}$, and so, by Theorem 6.2(1) and Theorem 4.13:

$$\begin{aligned} -\pi'(\tilde{\chi}_{S^1}(X))_{\mathbb{Q}} &= \text{PD}_X(\text{Eul}(\nu))_{\mathbb{Q}} = (\chi(\Sigma) - r)\{\gamma_0\}_{\mathbb{Q}} + \sum_{j=1}^r \frac{1}{\mu_j}\{\gamma_0\}_{\mathbb{Q}} \\ &= \left(\chi(\Sigma) + \sum_{j=1}^r \left(\frac{1}{\mu_j} - 1 \right) \right) \{\gamma_0\}_{\mathbb{Q}}. \quad \square \end{aligned}$$

Remark. In particular, $\tilde{\chi}_{S^1}(X)$ determines $\chi_V(\Sigma)$ whenever $\{\gamma_0\}$ has infinite order in $H_1(G)$.

From Proposition 6.6 we conclude an important theoretical fact about the connection between the invariants studied in this paper and the group $K_1(\mathbb{Z}G)$. Whenever one considers an invariant lying in $HH_1(\mathbb{Z}G)$ it is natural to ask if it is the image of a richer invariant lying in $K_1(\mathbb{Z}G)$ under the Dennis Trace map $DT: K_1(\mathbb{Z}G) \rightarrow HH_1(\mathbb{Z}G)$; we discuss this in a number of related contexts in [GN₄]. Proposition 6.6 shows that in the case of $\tilde{\chi}_1(X)$, with X a Seifert fiber space, the answer is often negative. In more detail: $\tilde{\mathcal{X}}_1''(X)(\gamma)$ is represented by $\sum_{j=1}^r \sum_{i=1}^{\mu_j-1} g_j^{-1} \otimes g_j g_j^{-i}$. If the Euler number is zero and X is aspherical and without boundary, then, by Proposition 6.6, $\{g_j\} \neq 0 \in H_1(Z(g_j))$ because $\mu_j \{g_j\} = \gamma_0 \neq 0 \in H_1(G)$. Thus if $r > 0$ there is a non-zero component of $\tilde{\mathcal{X}}_1(X)(\gamma)$ marked by a non-central element. Yet $\text{Wh}_1(G) = 0$ ([W]) and so it follows from Proposition 1.4 that $\tilde{\mathcal{X}}_1(X)(\gamma)$ does not lie in the image of the Dennis trace $DT: K_1(\mathbb{Z}G) \rightarrow HH_1(\mathbb{Z}G)$. Indeed DT extends to a Γ_X -module homomorphism $DT': \mathbb{Z}\Gamma_X \otimes K_1(\mathbb{Z}G) \rightarrow HH_1(\mathbb{Z}G)$ and $\tilde{\chi}_1(X)$ is not in the image of $DT'_*: H^1(\Gamma_X, \mathbb{Z}\Gamma_X \otimes K_1(\mathbb{Z}G)) \rightarrow H^1(\Gamma_X, HH_1(\mathbb{Z}G))$.

APPENDIX A. SOME CONSEQUENCES OF $\tilde{\chi}'_1(X) \neq 0$

In §2 we defined a cohomology class $\tilde{\chi}'_1(X) \in H^1(\Gamma_X, H_1(G) \otimes \mathcal{G}(X))$ which is a homotopy invariant of the space X (see Definition 2.5). Recall that $\Gamma_X = \pi_1(X^X, \text{id}_X)$ where X^X is the function space of self-maps of X and $\mathcal{G}(X)$ is the Gottlieb subgroup of $G \equiv \pi_1(X, x_0)$; see §2. In this Appendix we examine some group theoretic and topological consequences of the hypothesis $\tilde{\chi}'_1(X) \neq 0$.

Theorem A.1. *Suppose X is a finite aspherical CW complex and $\tilde{\chi}'_1(X) \neq 0$. Then $\Gamma_X \cong Z(G)$ is infinite cyclic.*

Proof. Since X is aspherical, by [Got] the evaluation at the basepoint $\eta: X^X \rightarrow X$ induces an isomorphism $\Gamma_X \cong Z(G)$. By hypothesis $\tilde{\chi}'_1(X) \neq 0$; in particular, the cohomology group $H^1(Z(G), H_1(G) \otimes_{\mathbb{Z}} \mathbb{Z}Z(G))$ is non-trivial. Since G is torsion free, Theorem 6.10 of

[DD, Ch.IV], implies that $Z(G)$ is either infinite cyclic or a non-trivial free product. The group $Z(G)$ is abelian and so cannot be a non-trivial free product. \square

In practice, we often have more information about $\tilde{\chi}'_1(X)$ (e.g. an explicit calculation of $\tilde{\chi}'_1(X)(\gamma)$ for certain $\gamma \in \Gamma_X$), leading to a sharper conclusion:

Theorem A.2. *Suppose X is a finite CW complex, $\gamma \in \Gamma_X$, and $\tilde{\chi}'_1(X)$ is represented by a derivation $\Delta: \Gamma_X \rightarrow HH_1(\mathbb{Z}G)'$ such that $\Delta(\gamma)$ has a non-zero entry in exactly one component. Then the Gottlieb subgroup, $\mathcal{G}(X)$, is cyclic and is generated by $\eta_{\#}(\gamma)$.*

Remark. Let μ be the composite homomorphism

$$HH_1(\mathbb{Z}G)' \cong H_1(G) \otimes \mathbb{Z}\mathcal{G}(X) \xrightarrow{\text{id} \otimes \varepsilon} H_1(G)$$

where $\varepsilon: \mathbb{Z}\mathcal{G}(X) \rightarrow \mathbb{Z}$ is the augmentation. The derivation Δ of Theorem A.2 is necessarily outer because $\mu(\Delta(\gamma)) \neq 0$ whereas $\mu(J(\gamma)) = 0$ for any inner derivation J . In particular $\tilde{\chi}'_1(X) \neq 0$; compare Theorem A.1.

Proof of Theorem A.2. We may assume that $\Delta(\gamma)$ is concentrated in the γ^0 -component (i.e., the component corresponding to the identity element). Since $\Delta(\gamma) \neq 0$, $\gamma \neq 1$. Let $\tau \in \Gamma_X$. Assuming $\eta_{\#}(\tau) \neq 1$, we will show $\eta_{\#}(\tau)$ is a power of $\eta_{\#}(\gamma)$. since $\tau\gamma = \gamma\tau$, we have $(1 - \tau)\Delta(\gamma) = (1 - \gamma)\Delta(\tau)$. For any $\lambda \in HH_1(\mathbb{Z}G)'$, the sum in $H_1(G)$ of the entries of $(1 - \gamma)\lambda$ in a $\langle \gamma \rangle$ -orbit of components of $HH_1(\mathbb{Z}G)'$ is obviously zero. Applying this observation with $\lambda = \Delta(\gamma)$, we find that $C(\eta_{\#}(\tau)^{-1}) = C(\eta_{\#}(\gamma)^{-k})$ for some k . Hence $\eta_{\#}(\tau) = \eta_{\#}(\gamma)^k$ because $\mathcal{G}(X) \equiv \eta_{\#}(\Gamma_X) \leq Z(G)$. \square

Corollary A.3. *If X in Theorem A.2 is aspherical, then $Z(G)$ is infinite cyclic and is generated by $\gamma \in \Gamma_X \cong Z(G)$.* \square

The hypothesis in Corollary A.3 that $\Delta(\gamma)$ be concentrated in one component is necessary in the following sense:

Proposition A.4. *Suppose $Z(G)$ is infinite cyclic with generator γ . Let*

$$\Delta: Z(G) \rightarrow HH_1(\mathbb{Z}G)' \cong \bigoplus_{\gamma^n \in Z(G)} H_1(G)$$

be a derivation. Then Δ differs by an inner derivation from a derivation Δ' such that $\Delta'(\gamma)$ has a non-zero entry in at most one component.

Proof. A derivation $Z(G) \equiv \langle \gamma \rangle \rightarrow HH_1(\mathbb{Z}G)'$ is freely determined by its value on γ . Write $\Delta(\gamma) = \sum_i \Delta_i(\gamma)$ where $\Delta_i(\gamma)$ has the same γ^i -component, say a_i , as $\Delta(\gamma)$ and all other components are zero. Define a derivation Δ'_i so that the γ^0 -component of $\Delta'_i(\gamma)$ is a_i and all other components are zero. Then $\Delta_i(\gamma) - \Delta'_i(\gamma) = (1 - \gamma^i)\Delta_i(\gamma) = (1 - \gamma)u_i$ where

$$u_i = \begin{cases} (1 + \gamma + \cdots + \gamma^{i-1})\Delta_i(\gamma) & \text{if } i > 0 \\ 0 & \text{if } i = 0 \\ (-\gamma^{-1} - \gamma^{-2} - \cdots - \gamma^i)\Delta_i(\gamma) & \text{if } i < 0. \end{cases}$$

Define $\Delta'(\gamma) = \sum_i \Delta'_i(\gamma)$. Then $\Delta(\gamma) - \Delta'(\gamma) = (1 - \gamma)(\sum_i u_i)$ and $\Delta'(\gamma)$ is concentrated in the γ^0 -component. \square

Remark A.5. Recall that $\chi_1(X) = \chi'_1(X) + \chi''_1(X)$ (see Definition 2.7). With a different and more difficult proof, Theorem 5.4 of [GN₃] provides the same conclusion as Theorem A.1 under the hypotheses that $\chi_1(X; \mathbb{Q}) \neq 0$ and G has the “Weak Bass Property over \mathbb{Q} ” (see [GN₃, §5]). Here, $\chi_1(X; \mathbb{Q}): \Gamma_X \rightarrow H_1(X; \mathbb{Q})$ is the composite of $\chi_1(X)$ with the coefficient homomorphism $H_1(X; \mathbb{Z}) \rightarrow H_1(X; \mathbb{Q})$. In Theorem A.1, the (possibly vacuous) K -theoretic hypothesis about the Weak Bass Property is absent. The hypothesis $\chi_1(X; \mathbb{Q}) \neq 0$ can be checked in many cases in view of the wide variety of equivalent definitions of $\chi_1(X)$ given in [GN₃]. However, it can happen that $\chi_1(X) = 0$ while $\tilde{\chi}'_1(X) \neq 0$; see Example 3.8 of [GN₃].

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